

CANDY SHARING AND DIFFERENCE BOXES:
EXAMPLES OF DISCRETE DYNAMICAL SYSTEMS

Utah Teachers' Math Circle

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Mathematics is like checkers in being suitable for the young, not too difficult, amusing, and without peril to the state.

— Plato (circa 400 BC)

1. CANDY SHARING

Consider the following game which was the subject of a problem from the 1962 Beijing Mathematical Olympiad. A number of students sit in a circle. Their teacher gives each of them some candy. Each student receives an even number of pieces. Then the teacher rings a bell, and each student passes half of his pile of candy to the student on his right. Each student who now has an odd number of pieces of candy is given an extra piece so that their pile again consists of an even number. Then the teacher rings a bell and the process begins again. The game stops when all students have the same amount of candy.

One can immediately ask if there is any reason to expect the game to end at all. If it does end, one can ask how many moves (that is, bell rings) it takes to reach the end of the game. (This will clearly depend on the initial distribution of candy.) And so on.

To get a feel for the game, let's consider a particular example where there are four students (Alice, Bert, Candice, and David) seated in a circle so that Bert is at Alice's right, Candice is at Bert's right, David is at Candice's right, and finally Alice is at David's right. Suppose that initially Alice has 8 pieces, Bert has 10, Candice has 2, and David has 0. The bell rings. Of Alice's 8, she gives away 4 to Bert, takes 0 from David, and is thus left with $8 - 4 + 0 = 4$. Meanwhile of Bert's 10, he gives 5 to Candice, receives 4 from Alice, and is left with $10 - 5 + 4 = 9$, which is odd. So he gets one additional one from the teacher to arrive at 10 again. Similarly Candice ends with $2 - 1 + 5 = 6$, and David ends with $0 - 0 + 1(+1) = 2$.

We can iterate the process and keep track of our results in the following table.

A	B	C	D
8	10	2	0
$4 = 8 - 4 + 0$	$10 = 10 - 5 + 4(+1)$	$6 = 2 - 1 + 5$	$2 = 0 - 0 + 1(+1)$
$4 = 4 - 2 + 1(+1)$	$8 = 10 - 5 + 2(+1)$	$8 = 6 - 3 + 5$	$4 = 2 - 1 + 3$
$4 = 4 - 2 + 2$	$6 = 8 - 4 + 2$	$8 = 8 - 4 + 4$	$6 = 4 - 2 + 4$
$6 = 4 - 2 + 3(+1)$	$6 = 6 - 3 + 2(+1)$	$8 = 8 - 4 + 3(+1)$	$8 = 6 - 3 + 4 + 1$
$8 = 6 - 3 + 4(+1)$	$6 = 6 - 3 + 3$	$8 = 8 - 4 + 3(+1)$	$8 = 8 - 4 + 4$
$8 = 8 - 4 + 4$	$8 = 6 - 3 + 4 + 1$	$8 = 8 - 4 + 3(+1)$	$8 = 8 - 4 + 4$
8	8	8	8

Thus the above game really did end in a finite number of steps. (It took only 6 rings of the bell.) Did we get lucky? What if we had started with a different initial distribution of candy? How about playing the game with 5 players? 10 players? 100?!

You should try a few examples to get warmed up.

The Beijing Olympiad problem asks for a proof that the game finishes in a finite number of steps no matter what the initial distribution of candy is. Here is a very clever way to see that. Let M denote the number of pieces in the possession of any one student at the outset. When the bell rings, a student with k pieces passes $k/2$ pieces to his right. The most he can receive from the person on his left is $M/2$ (since M was assumed maximal). Thus the most he can end up with is M (if he started with $k = M$ pieces and received $M/2$). Since M was assumed to be even, and he doesn't receive an additional piece from the teacher if he winds up with M . On the other hand, if he winds up with less than M , then the most he can obtain (even after possibly receiving an additional piece from the teacher) is indeed M . Thus we conclude that the game is bounded in the sense that *no student ever attains more than M pieces of candy*.

Now let's investigate the lower bound. Let m denote the smallest number of pieces in the possession of any one student at the outset. If a student begins with k pieces, when the bell rings he is left with an amount greater than or equal to $k/2 + m/2$ (since m was assumed to be minimal). The only way his pile could fail to grow after the bell rings is if he started with m pieces and the person on his left also started with m pieces. Thus if there are l people sitting in a row, each with m pieces, after the bell rings the last $l - 1$ will remain with m pieces while the first will have more than m pieces (unless everyone in the circle has m pieces, in which case the game would have already ended). Thus *the number of students with the smallest piles of candy decreases after each ringing of the bell*.

Since we have seen that there is an upper bound on the size of each pile of candy, the game must eventually stabilize with each student having the same amount of candy. QED.

The kind of argument used above is very powerful. We will have occasion to use a variant of it in the next section. But before leaving candy sharing, we study a more general class of games. Here is the setting. Suppose we have n players $1, 2, \dots, n$. Each player i is assigned a number p_i , and a list of n numbers

$$a_{i,1}, a_{i,2}, \dots, a_{i,n}$$

between 0 and p_i so that

$$a_{i,1} + a_{i,2} + \dots + a_{i,n} = p_i,$$

for each player i . The game starts with each player receiving a pile of candy so that the i th player's pile contains a number of pieces that is divisible by p_i . The teacher rings the bell. For each j , player i multiplies the number of pieces in his pile by $a_{i,j}/p_i$ and passes that amount to player j . The teacher then comes around and adds additional pieces to each player's pile so that each player i has an amount divisible by p_i . Then the bell rings again and the process begins anew.

For example, in our the game initially described above, there are n players, $p_i = 2$ for each of them and for $1 \leq i \leq n - 1$

$$a_{i,j} = \begin{cases} 1/2 & \text{if } j = i \text{ or } i + 1 \\ 0 & \text{otherwise} \end{cases}$$

while

$$a_{n,j} = \begin{cases} 1/2 & \text{if } j = n \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

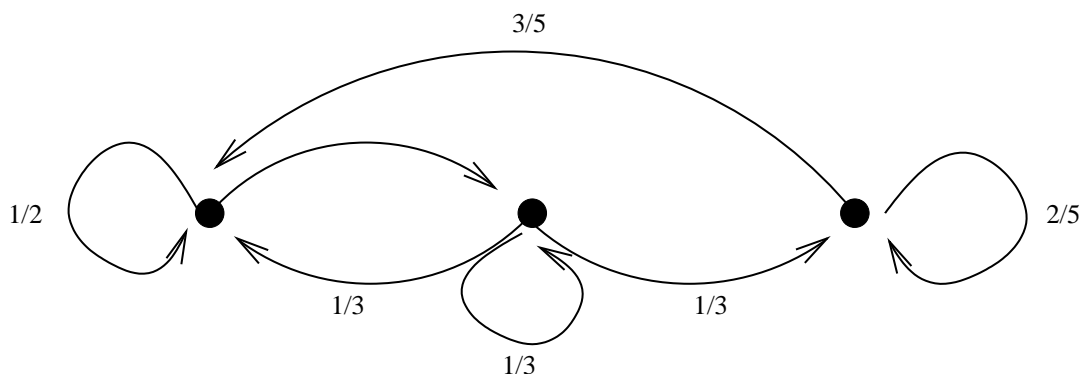


FIGURE 1. A graphical representation of a generalized three player game.

We can represent the general game graphically. As an example, consider the three person game where

$$p_1 = 2, p_2 = 3 \text{ and } p_3 = 5$$

and

$$\begin{aligned} a_{1,1} &= a_{1,2} = 1 \\ a_{2,1} &= a_{2,2} = a_{2,3} = 1 \\ a_{3,1} &= 3, \quad a_{3,3} = 2, \end{aligned}$$

while all other $a_{i,j}$'s are 0. We may represent this graphically as in Figure 1.

If Player 1 start with 6 pieces, Player 2 starts with 9, and Player 3 start with 10, we can play the game as follows.

Player 1	Player 2	Player 3
6	9	10
$12 = 6 - 3 + 3 + 6$	$6 = 9 - 6 + 3$	$10 = 10 - 6 + 3(+3)$
$14 = 12 - 6 + 2 + 6$	$9 = 6 - 4 + 6(+1)$	$10 = 10 - 6 + 2(+4)$
$16 = 14 - 7 + 6 + 3$	$12 = 9 - 6 + 7(+2)$	$10 = 10 - 6 + 3(+3)$
$18 = 16 - 8 + 6 + 4$	$12 = 12 - 8 + 8$	$10 = 10 - 6 + 4(+2)$
$20 = 18 - 9 + 6 + 4(+1)$	$15 = 12 - 8 + 9(+2)$	$10 = 10 - 6 + 4(+2)$
$22 = 20 - 10 + 6 + 5(+1)$	$15 = 15 - 10 + 10$	$10 - 6 + 5(+1)$
$22 = 22 - 11 + 6 + 5$	$18 = 15 - 10 + 11(+2)$	$10 = 10 - 6 + 5(+1)$
$24 = 22 - 11 + 6 + 6(+1)$	$18 = 18 - 12 + 11(+1)$	$10 = 10 - 6 + 6$
24	18	10

Problem. Describe a class of generalized games for which the above argument applies to show the game stabilizes.

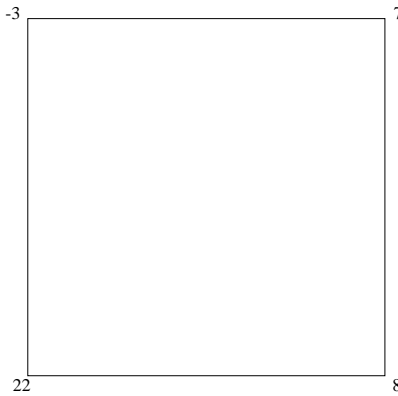
Warm-up exercise. Devise a three player game (involving all three players) of the above type which *never* finishes? Can you find a general construction of such games?

In fact, the argument we used for the original problem is not enough the more general case described above. (Similar but more subtle reasoning is needed.) I will simply describe the result here and refer you to the articles referenced below. Suppose one is given a generalized game represented graphically as above. Suppose that the game satisfies two requirements. First suppose that given any two players i and j , there is a directed path in the graph from i to j . Second suppose that there is at least one self-arrow; that is, suppose that there exists some player who does not give all of his candy away on each turn. Then one has the following theorem: the game stabilizes in a finite number of steps.

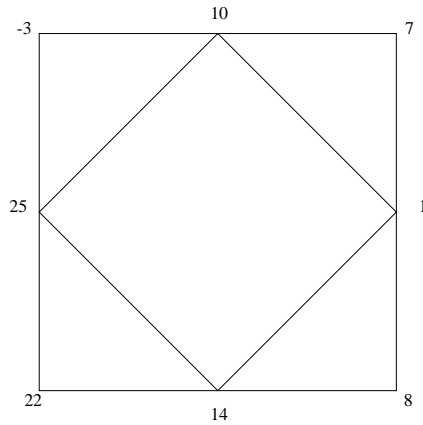
Notice that the original game satisfies the two hypotheses of the previous paragraph, and hence the general theorem implies that they eventually stabilize. It is an excellent challenge to try to prove the more general result without peeking at the reference given below.

2. DIFFERENCE BOXES

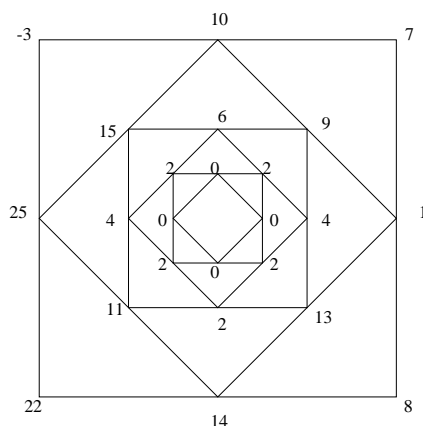
Consider the following game. Draw a square and label each of the corners with some whole number. At the midpoint of each square write the (positive) difference between the numbers at the corresponding vertices. Then draw a new square though the midpoints and repeat the process. For instance, consider the starting square



Then write in the positive differences at the midpoints and draw a new square:



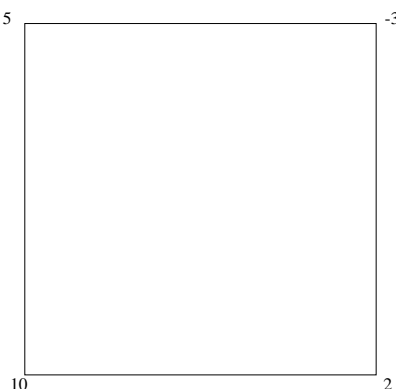
And then we repeat the process:



The remarkable observation is that we ended up at the square with all vertices labeled 0. Was that just luck? Was my choice of original square chosen in a special way so that we ended up with the zero box? What if we allow the starting numbers to be any real numbers (not just integers)? Does this change things? Given an integer k , can we find a starting box that reaches the zero box after exactly k steps? What about playing the game on pentagons? Hexagons? Cubes? (We are going to stick to boxes for today, but it's an excellent exercise to think about these other configurations.)

To get started, try some examples. Are there simple cases that you can show always converge? What if two of the adjacent starting values are the same?

In order to tackle the above questions, some notation is useful. Start with a diffy box and choose any corner of it. Beginning with that corner read off the numbers at the other corners moving in a clockwise fashion. This gives a list of 4 numbers, say a, b, c , and d . Then we label the diffy box $[a \ b \ c \ d]$. For instance if we start with the box



and choose the vertex 5, then we label the box $[5 \ -3 \ 2 \ 10]$. The label attached to a box is not well-defined in the following sense: if we started with the vertex 2, then we would label the box $[2 \ 10 \ 5 \ -3]$. But this ambiguity is harmless in practice.

As an example of this notation, look at our original example. Then we can describe the sequence of diffy boxes in our new notation as

$$[-3 \ 7 \ 8 \ 22] \rightarrow [10 \ 1 \ 14 \ 25] \rightarrow [9 \ 13 \ 11 \ 15] \rightarrow [4 \ 2 \ 4 \ 6] \rightarrow [2 \ 2 \ 2 \ 2] \rightarrow [0 \ 0 \ 0 \ 0].$$

As a matter of terminology, we say that the original box $[-3\ 7\ 8\ 22]$ has *longevity* 5 (since it takes five moves to reach the zero box).

Our next aim is to prove the following theorem:

Theorem. *If a, b, c and d are integers, then the diffy box $[a\ b\ c\ d]$ has finite longevity.*

The key tool in the proof of the theorem is a notion of the “size” of a diffy box:

Definition. Given a diffy box $[a\ b\ c\ d]$, define the size of $[a\ b\ c\ d]$, denoted $||[a\ b\ c\ d]||$ to be the largest difference among (not necessarily adjacent) pairs of vertices. For example, $||[-3\ 7\ 8\ 22]|| = 25$ since $22 - (-3) = 25$ is the largest difference among $-3, 7, 8, 22$.

As an example, we compute the sizes of the various boxes in the first example considered above. As before, may write that sequence of boxes as

$$[-3\ 7\ 8\ 22] \rightarrow [10\ 1\ 14\ 25] \rightarrow [9\ 13\ 11\ 15] \rightarrow [4\ 2\ 4\ 6] \rightarrow [2\ 2\ 2\ 2] \rightarrow [0\ 0\ 0\ 0].$$

The corresponding sizes are easy to write down,

$$25 \rightarrow 24 \rightarrow 6 \rightarrow 4 \rightarrow 0 \rightarrow 0.$$

The key observation is that the sizes in this sequence never increase. In fact, they strictly decrease until we reached the last two terms (where the sizes were both zero). This should remind you of the situation we encountered with the candy sharing game!

Here is the general result we need:

Lemma. *Given a pair of diffy boxes $B_1 \rightarrow B_2$, then $|B_1| \geq |B_2|$. If no two adjacent corners of B_1 have the same value, then $|B_1| > |B_2|$.*

Proof. Let $a < b < c < d$ denote the four entries of B_1 arranged in increasing order. Then $|B_1| = d - a$. Meanwhile the four entries of B_2 are all between 0 and $d - a$. So the size of B_2 is at most $d - a$. This shows that $|B_1| \geq |B_2|$. When can $|B_1| = |B_2|$? That is, when does $|B_2| = d - a$? Since all entries of B_2 are between 0 and $d - a$, $|B_2| = d - a$ only if the entries 0 and $d - a$ are adjacent in B_2 . But if 0 appears in B_2 , then two adjacent vertices of $|B_1|$ must have the same value. This proves the second assertion of the lemma. \square

Now we can return to the theorem above and try to prove it. We start with any diffy box, say B_1 , and start performing our difference operation to get a sequence $B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \dots$. At each step the size is shrinking: $|B_1| \geq |B_2| \geq |B_3| \geq \dots$. Let k denote the smallest integer such that $|B_k| = |B_{k+1}|$ (instead of $|B_k| > |B_{k+1}|$). Such a k always exists. (Why?!) If $|B_k| = 0$, then all entries of B_k are the same, B_{k+1} is the zero box, and the longevity of B_1 is k . On the other hand, suppose that $|B_k| \neq 0$. Then it looks like our argument might be stuck. To finish it we need to know one final exercise (which you may have discovered in your consideration of examples above):

Exercise. Suppose $B = [a\ a\ x\ y]$. Show that the longevity $l(B)$ is less than or equal to 6. (This exercise completes the proof of the theorem.)

The previous theorem handles that case of *integer* diffy boxes. (Where in the proof did we use the hypothesis that all numbers involved were integers?) So that leads us to ask about noninteger diffy boxes. This is quite a bit more subtle. I refer you to the references given below.

3. REFERENCES

A good reference for the material on candy sharing is the article “Candy Sharing,” by G. Iba and J. Tanton, *American Math. Monthly*, volume 110 (2003), no. 1, pp. 25–35. A good reference for the material on difference boxes is “The Convergence of Difference Boxes,” by A. Behn, C. Kribs-Zaleta, and V. Ponomarenko, *American Math. Monthly*, volume 112 (1995), pp. 426–438. Much of the notation and terminology for difference boxes which I use is borrowed from that paper. An exhaustive list of references for difference boxes can be found at <http://mathed.uta.edu/~kribs/diffy.html>. You may also wish to consult my Math Circle notes on the subject: <http://www.math.utah.edu/mathcircle/notes/diffybox.pdf> and <http://www.math.utah.edu/mathcircle/notes/candy.pdf>.