Solutions
UTAH STATE MATH CONTEST
AT THE
UNIVERSITY OF UTAH
MARCH 16, 2006
GRADES 10-12

1. Given the array on the right and the rules below, how many different paths can you take to spell UTAHMATH?

- Begin at the top.
- Move only down.
- For each move. go to one of the one or two letters directly below the letter you are on.

U_2
$T_2\ T_2$
$A_2\ A_2\ A_2$
$H_2\ H_2\ H_2\ H_2$
$M_1\ M_2\ M_2\ M_2\ M_1$
$A_1\ A_2\ A_2\ A_1$
$T_1\ T_2\ T_1$
$H\ \ H$

Use Pascal's triangle here.
I wrote # of choices next to each spot.

a. 2 b. 10 c. 20 d. 35 e. 70

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

This corresponds to the circled M above. It means we have 4 ways to get to M. ⇒ There are 35 ways to get to bottom right H and another 35 ways to get to bottom left H.

2. Between which two integers does $\sqrt{2006}$ fall?

a. 42, 43 b. 43, 44 c. 44, 45 d. 45, 46 e. 46, 47

We know $40^2 = 1600$ and $50^2 = 2500$.

Try the choices.

$$\begin{array}{r} 44 \\ \times 44 \\ \hline 176 \\ 176\ \ \\ \hline 1936 \end{array} \qquad \begin{array}{r} 45 \\ \times 45 \\ \hline 225 \\ 180\ \ \\ \hline 2025 \end{array}$$

$1936 < 2006 < 2025 \Rightarrow 44 < \sqrt{2006} < 45$

3. What is the greatest common factor of 10^9 and $25!$?

a. 25 b. 10^9 c. $2^3 x 10^6$ d. 10^6 e. 10^5

$10^9 = 2^9 5^9$

$25! = (5^2)(2^3 \cdot 3)(23)(2 \cdot 11)(3 \cdot 7)(2^2 \cdot 5)(19)(2 \cdot 3^2)(17)(2^4)(3 \cdot 5)(2 \cdot 7)$
$(13)(2^2 \cdot 3)(11)(2 \cdot 5)(3^2)(2^3)(7)(2 \cdot 3)(5)(2^2)(3)(2)(1)$

25! has 2^9 as a factor and 5^6 $\Rightarrow$ $GCF = 2^9 5^6 = 10^6(2^3)$

4. Students in Mr. Zeno's class had these scores on a test: 85, 84, 69, 91, 80, 77, 92, 96, 76. Somehow he had forgotten to record Dori's score, but he knew the class mean with her score was 83. Determine Dori's score.

a. 80 b. 83 c. 90 d. 84 e. 85

$$\frac{85+84+69+91+80+77+92+96+76+x}{10} = 83$$

$750 + x = 830$

$x = 80$

5. There are 26 students in a class , including exactly 12 girls and 20 sophomores. What is the minimum possible number of sophomore girls in the class?

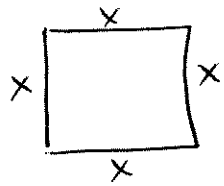
a. 12 b. 10 c. 8 d. 6 e. There is not enough information.

$b + g = 26$ $g = 12$ $\Rightarrow$ $b = 14$

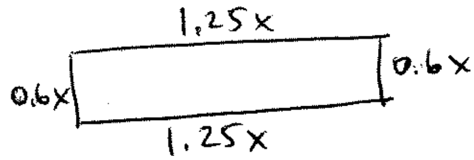
Since there are 14 boys, but 20 sophomores, the minimum # girls $= 20 - 14 = 6$.

6. Two opposite sides of a square are increased by 25% and the other two are decreased by 40%. What is the percent decrease in the area of the resulting rectangle?

a. 2.25% b. 15% c. 25% d. 40% e. 65%



$A = x^2$



$A = (1.25x)(0.6x) = 0.75x^2$

i.e. a decrease of 25%

7. $\left(\sqrt{a+x} - \sqrt{x-a}\right)\left(\sqrt{a+x} + \sqrt{x-a}\right)$ equals:

a. 0 b. 2x c. 2x+2a d. 2a e. $\sqrt{a+x} - \sqrt{x-a}$

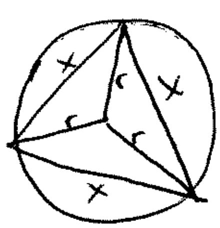
$$\left(\sqrt{a+x} - \sqrt{x-a}\right)\left(\sqrt{a+x} + \sqrt{x-a}\right) = a + x + \sqrt{(a+x)(x-a)} - \sqrt{(x-a)(a+x)} - (x-a)$$

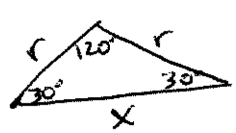
$$= a + x - x + a$$

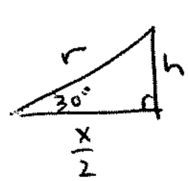
$$= a + a = 2a$$

8. An equilateral triangle is inscribed in a circle. What is the ratio of the area of the triangle to the area of the circle?

a. $\frac{\sqrt{3}}{\pi}$ b. $\frac{2}{\pi}$ c. $\frac{2\sqrt{3}}{\pi}$ d. $\frac{3\sqrt{3}}{4\pi}$ e. There is not enough information.







$\cos 30° = \frac{\frac{x}{2}}{r} \Rightarrow x = \sqrt{3}\, r$

$\sin 30° = \frac{h}{r} \Rightarrow h = \frac{1}{2} r$

$A_\Delta = 3\left(\frac{1}{2} x h\right) = \frac{3}{2}\left(\sqrt{3}\, r\right)\left(\frac{r}{2}\right) = \frac{3\sqrt{3}}{4} r^2$

$A_\circ = \pi r^2$

$$\frac{A_\Delta}{A_\circ} = \frac{\frac{3\sqrt{3}}{4} r^2}{\pi r^2} = \frac{3\sqrt{3}}{4\pi}$$

9. Convert $1.2\overline{6}$ to a fraction.

a. $1\frac{1}{3}$ b. $1\frac{26}{99}$ c. $1\frac{13}{50}$ d. $1\frac{5}{33}$ e. $1\frac{4}{15}$

let $n = 1.2\overline{6}$

$$1000n = 1266.\overline{6}$$
$$-100n = 126.\overline{6}$$
$$900n = 1140$$
$$n = \frac{1140}{900} = \frac{114}{90} = \frac{38}{30} = \frac{19}{15} = 1\frac{4}{15}$$

10. A grocer wants to sell a mixture of jelly beans and chocolate eggs for \$4.75 per pound. If the jelly beans cost \$6.00 per pound and the chocolate eggs cost \$4.00 a pound, how many pounds of jelly beans will we need to make 100 lbs of the mixture?

a. 37.5 lbs b. 62.5 lbs c. 60 lbs d. 7.5 lbs e. 15 lbs

$j + c = 100$

$6j + 4c = 4.75(100) \Rightarrow 6(100-c) + 4c = 475$

$600 - 2c = 475$

$125 = 2c$

$62.5 = c \Rightarrow j = 37.5$ lbs

11. A store has a five-day sale where all merchandise is discounted by 1/3 on the first day. Beginning on the second day and each day thereafter, they take an additional 10% off the previous day's price. What will you pay for an item on the third day if it cost \$120 before the sale began? (Round your answer to the nearest whole dollar.)

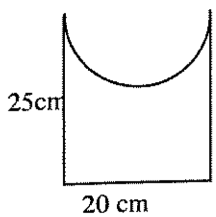
a. \$58 b. \$55 c. \$65 d. \$64 e. \$56

1st day $\Rightarrow$ price $\$120\left(\frac{2}{3}\right) = \$80$

2nd day $\Rightarrow$ price is $0.9(80) = \$72$

3rd day $\Rightarrow$ price is $0.9(72) = \$64.80 \simeq \$65$

12.



In this drawing, a semicircle has been cut out of a rectangle with the given measurements. Which of these numbers is closest to the area of the enclosed shape?

a. 100 cm^2 b. 130 cm^2 c. 200 cm^2 d. 350 cm^2 e. 470 cm^2

$A = A_{\square} - A_{\smile} = 20(25) - \frac{1}{2}\left(\pi(10^2)\right) = 500 - 50\pi$

$\simeq 500 - 150 = 350\ cm^2$

13. Let $y_1 = f(x) = \frac{x+1}{x-1}$; $y_2 = f(y_1)$; $y_3 = f(y_2)$; $y_n = f(y_{n-1})$, for n = 1,2,3,4,…

Find y_{100}

a. x b. $\frac{x+1}{x-1}$ c. $\frac{x+100}{x-100}$ d. $\frac{100x+1}{100x-1}$ e. None of these

$y_1 = \frac{x+1}{x-1}$

$y_2 = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{x+1+x-1}{x+1-(x-1)} = \frac{2x}{2} = x$

$y_3 = \frac{x+1}{x-1} = y_1$

$y_4 = y_2 \dots y_{100} = y_2 = x$

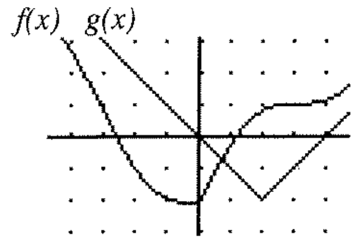
14. Solve this cryptarithm, where each letter represents a digit and no digit is represented by two different letters. The letter L represents what digit?

$$\begin{array}{r} F\ E\ L\ T \\ +\ M\ I\ C\ E \\ \hline M\ I\ L\ E\ S \end{array}$$

$$\begin{array}{r} 9\ 5\ 6\ 7 \\ +\ 1\ 0\ 8\ 5 \\ \hline 1\ 0\ 6\ 5\ 2 \end{array}$$

a. 3 b. 4 c. 6 d. 8 e. 9

15. Given $f(x)$ *and* $g(x)$ as pictured at the right, determine $(f \circ g)(3)$



a. 0 b. -1 c. 1 d. -2 e. 2

$(f \circ g)(3) = f(g(3)) = f(-1) = -2$

16. What is the remainder when 3^{3333} is divided by 5?

a. 0 b. 1 c. 2 d. 3 e. 4

n	ones digit of 3^n
0	1
1	3
2	9
3	7
4	1
5	3
6	9
7	7

So ones digit of 3^n repeats every 4^{th} time.

$3^{3333} = 3^{4(833)+1} = \underbrace{(3^4)^{833}}_{\text{ends in } 1} 3 \Rightarrow$ ones digit is 3

When divided by 5, the remainder will be 3.

17. Suppose a bag contains the six letters of the word "booboo." If you take one letter out of the bag at a time and line them up from left to right, what is the probability that you will spell the word "booboo"?

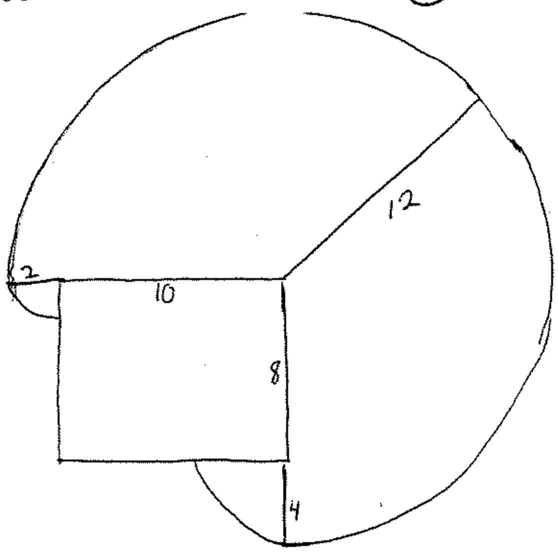
a. 1/4 b. 1/8 c. 1/15 d. 1/16 e. 1/30

There are 6! ways of combining the letters in "booboo." And since we have repeated letters in "booboo" (i.e. we can't differentiate between the b's or the o's), there are $2 \cdot 4!$ ways to get that word.

$P(\text{booboo}) = \frac{2(4!)}{6!} = \frac{2}{6 \cdot 5} = \frac{1}{15}$

18. A goat is tied to the corner of a building whose base is 8 ft by 10 ft with a twelve foot rope. How many square feet of grass can he reach to eat?

a. 108π b. 110π c. 113π d. 116π e. 144π



$\frac{3}{4}$ of r = 12 circle
$\frac{1}{4}$ of r = 2 circle
$\frac{1}{4}$ of r = 4 circle

$\frac{3}{4}(144\pi)+\frac{1}{4}(4\pi)+\frac{1}{4}(16\pi)$

$108\pi+\pi+4\pi = 113\pi$

19. Which of these is a perfect square in every possible base, b, for that number? (Eg. 321_b is only a number when b>3)

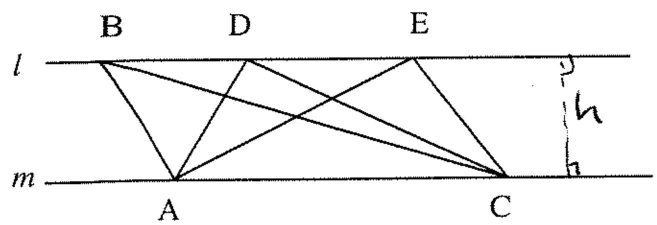
a. 36_b b. 81_b c. 144_b d. 225_b e. none of these

$1(b^2)+4(b)+4$

$= b^2+4b+4$

$=(b+2)^2$ perfect square for all b.

20. Given that the lines l and m are parallel, which of the three triangles has the greatest area, Δ ABC, ΔADC, ΔAEC?



a. ΔABC b. ΔADC c. ΔAEC d. The areas are all the same e. There is not enough information

The triangles all have the same base and height, h.

21. Evaluate this determinant:

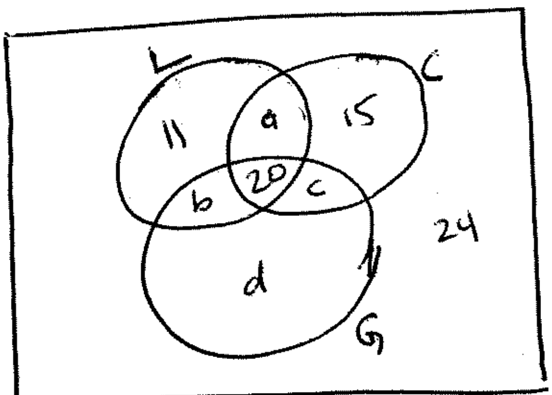
$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 5 & 0 \\ 1 & 1 & 2 & 2 & 5 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 5 & 0 \\ 1 & 2 & 2 & 5 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 3 & 5 & 0 \\ 2 & 2 & 5 & 8 \end{vmatrix} = -2\begin{vmatrix} 3 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 5 & 8 \end{vmatrix}$$

a. 240 b. 0 c. –80 d. 80 e. none of these.

$$= -2(3)\begin{vmatrix} 5 & 0 \\ 5 & 8 \end{vmatrix} = -6(40-0) = -240$$

22. In a survey of 115 people, only 20 liked all 3 candies: licorice, chocolate and gumdrops. Twenty-four did not like any of the candy, 15 liked only chocolate, 41 disliked chocolate but liked at least one of the other two kinds of candy. If 27 liked exactly 2 of the 3 candies, 11 liked only licorice and 59 liked gumdrops, how many liked chocolate and licorice, but not gumdrops?

a. 4   b. 6   c. 26   d. 32   e. Not enough information



115 total
✓ 20 $L \cap C \cap G$
✓ 24 none
✓ 15 C only
41 $L \cup G - C$
27 only 2
✓ 11 L only
59 G

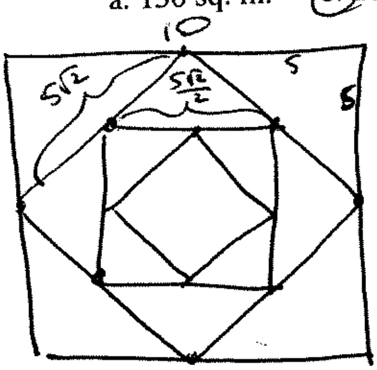
① $a+b+c+d+70=115$
② $b+d+11=41 \Rightarrow b+d=30$
③ $a+b+c=27$
④ $b+c+d+20=59 \Rightarrow b+c+d=39$

① $a+39+70=115 \rightarrow a=6$
④ $30+c=39 \rightarrow c=9$
③ $6+b+9=27 \rightarrow b=12$
② $12+d=30 \rightarrow d=18$

a = # who liked C + L but not G

23. A square has sides of ten inches each. A second square is inscribed in the original square by connecting the midpoints of the sides. Connecting the midpoints of the sides of the second square then forms a third square. This process is continued endlessly. What is the sum of the areas of the infinite sequence of squares?

a. 150 sq. in.   b. 200 sq. in.   c. $(150+50\sqrt{2})$ sq. in.   d. 300 sq. in.   e. The sum is infinite.



10

$$\sum_{i=0}^{\infty} A_i = 10^2 + \left(\frac{\sqrt{2}}{2}(10)\right)^2 + \left(\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}(10)\right)\right)^2 + \left(\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}(10)\right)\right)\right)^2 + \cdots + \left(\frac{\sqrt{2}}{2}\right)^{2n} 10^2 + \cdots$$

$$= 100\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots\right)$$

$$= 100 \sum_{i=0}^{\infty} \frac{1}{2^i} = 100\left(1 + \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i\right) = 100\left(1 + \frac{1/2}{1-1/2}\right)$$

$$= 100(1+1)$$
$$= 200$$

24. Which vector is NOT a linear combination of $\langle 1,0,-1\rangle$ and $\langle 1,1,2\rangle$? Assume scalars are in $\Re$.

a. $\langle -1,1,4\rangle$ b. $\langle 0,0,0\rangle$ c. $\langle 0,-1,-3\rangle$ d. $\langle 2,1,2\rangle$ e. $\langle 3\pi, \pi, 0\rangle$

A linear combo of $\langle 1,0,-1\rangle$ & $\langle 1,1,2\rangle$ looks like $a\langle 1,0,-1\rangle + b\langle 1,1,2\rangle$ for some $a,b \in \mathbb{R}$.

$= \langle a+b, b, 2b-a\rangle$

check all answers. For example, $\langle -1,1,4\rangle \stackrel{?}{=} \langle a+b, b, 2b-a\rangle$ ✓

$b=1 \Rightarrow a+1=-1$, $a=-2$; $2b-a = 2+2 = 4$

The only one that doesn't work is $\langle 2,1,2\rangle$.

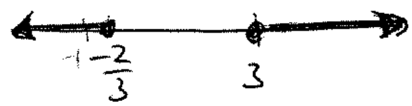
25. How many of these intervals are subsets of the domain of $f(x) = \dfrac{\sqrt{3x^2 - 7x - 6}}{x - 4}$?

(-∞,-1), (-7,0), [π,4), [3,10), (8,+∞)
yes no yes no yes

a. 0 b. 1 c. 2 d. 3 e. 4

domain of f(x) is $3x^2 - 7x - 6 \geq 0$ + $x \neq 4$

$(3x+2)(x-3) \geq 0$



but $x \neq 4$

domain: $x \in (-\infty, -\frac{2}{3}] \cup [3,4) \cup (4,\infty)$

26. Ten writers covering a basketball league vote for the Most Valuable Player of the league by listing their top three choices. A first place vote earns five points, a second place vote earns three points, and a third place vote earns one point. Janey Joyful received 37 points. How many writers listed her on their ballots?

a. 7 b. 8 c. 9 d. 10 e. There is not enough information to determine this.

$5f + 3s + t = 37$ and $f+s+t \leq 10$

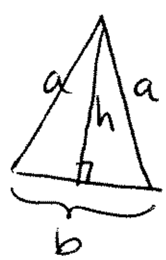
The only choices for f, s, t that satisfy $f+s+t \leq 10$ are

f	s	t
7	0	2
6	2	1
5	4	0

Each row of table adds to 9 total votes.

27. Of all the isosceles triangles with perimeter p, what is the ratio of the height to the perimeter of the one with the largest area?

a. 0 b. 1/3 c. $\dfrac{1}{\sqrt{6}}$ d. $\dfrac{1}{2\sqrt{3}}$ e. $\dfrac{1}{2\sqrt{2}}$



$2a + b = p \Rightarrow b = p - 2a$

$A = \frac{1}{2}bh \longrightarrow A = \frac{1}{2}(p-2a)\sqrt{ap - \frac{1}{4}p^2}$

$\left(\frac{b}{2}\right)^2 + h^2 = a^2$

$\Rightarrow h = \sqrt{a^2 - \frac{b^2}{4}}$

Since $b = p - 2a$, $h = \sqrt{ap - \frac{1}{4}p^2}$

Find $\frac{dA}{da}$ + set $\frac{dA}{da} = 0$, to get max A.

Solve for a to get $a = \frac{p}{3}$

$\Rightarrow h = \sqrt{\frac{p^2}{3} - \frac{p^2}{4}} = \frac{p}{\sqrt{12}} = \frac{p}{2\sqrt{3}}$

And $\frac{h}{p} = \frac{1}{2\sqrt{3}}$

28. Compute the following limit: $\lim_{n\to\infty}\int_0^1 x^n \cos(nx)dx$

a. 0 b. 1 c. $\pi/2$ d. π e. does not exist.

$$-1 \le \cos(nx) \le 1$$
$$\int_0^1 -x^n dx \le \int_0^1 x^n \cos(nx)dx \le \int_0^1 x^n dx$$
$$\frac{-1}{n+1} \le \int_0^1 x^n \cos(nx)\,dx \le \frac{1}{n+1}$$
$$\lim_{n\to\infty} \frac{-1}{n+1} \le \lim_{n\to\infty} \int_0^1 x^n \cos(nx)dx \le \lim_{n\to\infty} \frac{1}{n+1}$$
$$0 \le \lim_{n\to\infty} \int_0^1 x^n \cos(nx)dx \le 0$$

29. Suppose it takes h minutes to fill a bath tub using the hot water faucet and c minutes to fill the same tub using the cold water faucet. Starting with an empty tub, the hot water faucet is turned on and then after 1 minute, the cold water faucet is also turned on. How long will it take to fill the tub?

a. $\frac{h+(c-1)}{2}$ b. $\frac{h(c+1)}{2}$ c. $\frac{h(c+1)}{h+c}$ d. $\frac{hc}{c+h}$ e. $\frac{hc}{h+c}-1$

After 1 minute, there's only $1-\frac{1}{h}$ of job to finish.

And $\frac{1}{h}+\frac{1}{c}=\frac{1}{t} \Rightarrow t=\frac{hc}{h+c}$ = time it takes to do entire job together

$\Rightarrow$ total time $= \underbrace{1}_{\text{1 min}} + \underbrace{\left(1-\frac{1}{h}\right)\left(\frac{hc}{h+c}\right)}_{\text{rest of job together}} = \frac{h+c+(h-1)c}{h+c}$

$$= \frac{h+hc}{h+c} = \frac{h(1+c)}{h+c}$$

30. Find x if $x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}=2$

a. $\pm\sqrt{2}$ b. $2\pm\sqrt{2}$ c. $\frac{1}{2}$ d. $2\pm 2i\sqrt{2}$ e. $2-\sqrt{2}$

$$x+\sqrt{x+\sqrt{x+\cdots}} = 2$$
$$\left(\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}\right)^2 = (2-x)^2$$
$$x+\sqrt{x+\sqrt{x+\cdots}} = 4-4x+x^2$$

but $x+\sqrt{x+\sqrt{x+\cdots}} = 2$

$$\Rightarrow 2 = 4-4x+x^2$$
$$0 = x^2-4x+2$$
$$x = \frac{4\pm\sqrt{16-4(2)}}{2} = 2\pm\sqrt{2}$$

But $(2+\sqrt{2})+\sqrt{(2+\sqrt{2})+\sqrt{2+\sqrt{2}+\cdots}} = 2$

means $\sqrt{2}+\sqrt{2+\sqrt{2}+\sqrt{2+\sqrt{2}+\cdots}} = 0$

i.e. $\sqrt{2}+\underbrace{\text{something}}_{\text{positive}} = 0$

which can't happen

So, $2-\sqrt{2}$ is only answer.