Calculus II
Practice Problems 4: Answers

1. Integrate $\int (\ln x)^2 dx$.

Answer. We integrate by parts, using $u = (\ln x)^2,\ du = 2\ln x dx/x,\ dv = dx,\ v = x$:

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x dx \ .$$

As we saw in example 9 (another integration by parts):

$$\int \ln x dx = x\ln x - x + C \ , \quad \text{so}$$

(1)
$$\int (\ln x)^2 dx = x(\ln x)^2 - 2(x\ln x - x) + C \ .$$

---

2. Integrate $\int x^2 \ln x dx$.

Answer. Let $u = \ln x,\ du = dx/x,\ dv = x^2 dx,\ v = x^3/3$:

$$\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C \ .$$

---

3. Integrate $\int \arccos x dx$.

Answer. Let $u = \arccos x,\ du = -dx/\sqrt{1-x^2},\ dv = dx,\ v = x$:

(2)
$$\int \arccos x dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \ .$$

This last we integrate by the substitution $w = 1 - x^2,\ dw = -2x dx$:

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int w^{-1/2} dw = -w^{1/2} + C \ .$$

Putting this back in (2) we obtain

$$\int \arccos x dx = x \arccos x - \sqrt{1-x^2} + C \ .$$

---

4. If the region in the first quadrant bounded by the curves $y = 1,\ y = e^x$ and $x = 1$ is rotated about the y-axis, what is the volume of the resulting solid?

Answer. The region is drawn in figure 1. One can sweep out the volume in the y-direction, using the method of washers, or in the x direction, using the method of shells.

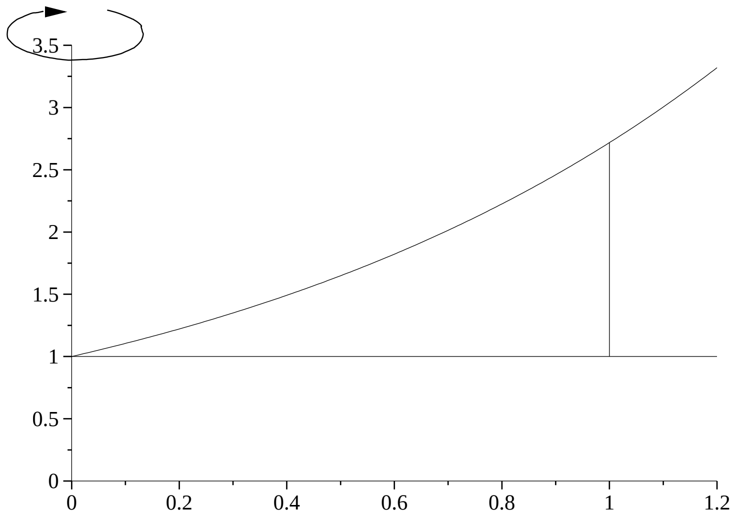

Figure 1

Washers. Here, sweeping in the y direction, the differential of volume is $dV = \pi(R^2 - r^2)dy$. The larger radius is $R = 1$, and the smaller radius is $r = \ln y$. Thus

$$Volume = \pi \int_1^e (1 - (\ln y)^2)dy = \pi(y - y(\ln y)^2 + 2y\ln y - 2y)\Big|_1^e$$

using the solution to problem 1 above (equation (1)). Then

$$Volume \ = \pi(e - e + 2e - 2e - (1-2)) = \pi \ .$$

Shells. Now, we sweep out along the x-axis, and the differential of volume is $dV = 2\pi rhdx$. The radius is x, and the height of the shell is $e^x - 1$. Thus

$$Volume = 2\pi \int_0^1 x(e^x - 1)dx = 2\pi[xe^x - e^x - \frac{x^2}{2}]\Big|_0^1$$

using the formula obtained in example 5 of the Notes. We get

$$Volume = 2\pi(e - e - \frac{1}{2} - (0-1)) = \pi \ .$$

---

5. Integrate $\int \sec^3 x dx$.

Answer. First, we use the identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \sec^3 u du = \int (\tan^2 x + 1)\sec x dx = \int \tan^2 x \sec x dx + \int \sec x dx \ . \tag{3}$$

The last integral was computed in example 3 of the notes, so we concentrate on the first integral. If we write $\tan^2 x \sec x dx = \tan x(\sec x \tan x)dx$, then we can integrate by parts with the substitution $u = \tan x,\ du = \sec^2 x dx,\ dv = \sec x \tan x dx,\ v = \sec x$. Then

$$\int \tan^2 x \sec x dx = \sec x \tan x - \int \sec^3 x dx \ .$$

It appears we're back where we started, but not exactly. Substitute this in (3) to get:

$$\int \sec^3 u du = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \ .$$

Moving the second term to the left hand side, and dividing by 2, we have the result

$$\int \sec^3 u du = \frac{1}{2}(\sec x \tan x + \int \sec x dx) = \frac{1}{2}(\sec x \tan x) + \frac{1}{4}\ln(\frac{1+\sin x}{1-\sin x}) + C \ .$$

Incicentally, the expression found in example 3 of the Notes for $\int \sec x dx$ is not the usual one, but is equivalent to the formula found in most integral tables:

$$\int \sec x dx = \ln|\sec x + \tan x| + C \ , \tag{4}$$

using this sequence of identities:

$$\ln(\frac{1+\sin x}{1-\sin x}) = \ln(\frac{(1+\sin x)^2}{1-\sin^2 x}) = \ln(\frac{1+\sin x}{\cos x})^2$$

$$= 2\ln|\frac{1+\sin x}{\cos x}| = 2\ln|\sec x + \tan x| \ .$$

Finally, this gives the answer to our problem:

$$\int \sec^3 u du = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C \ . \tag{5}$$

---

6. Integrate (a) $\int \frac{(x+1)dx}{x(x+3)}$ (b) $\int \frac{(x+1)dx}{x^2(x+3)}$

Answer. For part (a), we seek constants A, B such that

$$\frac{(x+1)}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} \ .$$

Put the right hand side over a common denominator and equate numerators, getting $x+1 = A(x+3)+Bx$. Now substituting $x = 0$, we get $1 = 3A$, so $A = 1/3$. Substitute $x = -3$ to get $-3+1 = B(-3)$, so $B = 2/3$. Thus

$$\int \frac{(x+1)dx}{x(x+3)} = \frac{1}{3}\int \frac{dx}{x} + \frac{2}{3}\int \frac{dx}{x+3} = \frac{1}{3}(\ln x) + \frac{2}{3}\ln(x+3) + C = \frac{1}{3}\ln(x(x+3)^2) + C \ .$$

For part (b), we seek constants A, B, C such that

$$\frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \ .$$

Put the right hand side over a common denominator and equate numerators, getting $x+1 = Ax(x+3)+B(x+3)+Cx^2$. Now substituting $x = 0$, we get $1 = 3B$, so $B = 1/3$. Substitute $x = -3$ to get $-3+1 = C(9)$, so $C = -2/9$. That uses up the roots, so to find A we have to equate coeffiecients. Equating the coefficients of x^2 we get $0 = A + C$, so $A = 2/9$. Now, we can integrate:

$$\int \frac{(x+1)dx}{x^2(x+3)} = \int \frac{2/9}{x}dx + \int \frac{1/3}{x^2}dx - \int \frac{2/9}{x+3}dx = \frac{2}{9}\ln x - \frac{1}{3}x^{-1} - \frac{2}{9}\ln(x+3) + C \ .$$

---

7. Integrate $\int \frac{dx}{(x-1)(x+2)^2}$.

Answer. We seek constants A, B, C such that

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \,.$$

Put the right hand side over a common denominator and equate numerators, getting $1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$. Substitute $x = 1$: $1 = A(9)$, so $A = 1/9$. Substitute $x = -2$: $1 = C(-3)$, so $C = -1/3$. Now equate coefficients of x^2: $0 = A + B$, so $B = -1/9$. Thus

$$\int \frac{dx}{(x-1)(x+2)^2} = \int \frac{(1/9)dx}{x-1} - \frac{(1/9)dx}{x+2} - \int \frac{(1/3)dx}{(x+2)^2} =$$

$$= \frac{1}{9}\ln\frac{x-1}{x+2} + \frac{1}{3}(x+2)^{-1} + C \,.$$

---

8. Integrate $\int \frac{(x^2-1)dx}{(x^2+1)(x+3)}$.

Answer. We seek constants A, B, C such that

$$\frac{x^2-1}{(x^2+1)(x+3)} = \frac{A}{x^2+1} + \frac{Bx}{x^2+1} + \frac{C}{(x+3)} \,.$$

Put the right hand side over a common denominator and equate numerators, getting $x^2 - 1 = A(x+3) + Bx(x+3) + C(x^2+1)$. Substitute $x = -3$ to get $(-3)^2 - 1 = C((-3)^2 + 1)$, giving $C = 4/5$. To find A and B we must equate coefficients. For the constant term this gives $-1 = 3A + C$, so $A = -3/5$. Equating coefficients of x gives us $0 = A + 3B$, so $B = -A/3 = 1/5$. Thus

$$\int \frac{(x^2-1)dx}{(x^2+1)(x+3)} = -\frac{3}{5}\int \frac{dx}{x^2+1} + \frac{1}{5}\int \frac{xdx}{x^2+1} + \frac{4}{5}\int \frac{dx}{x+3}$$

$$= -\frac{3}{5}\arctan x + \frac{1}{10}\ln(x^2+1) + \frac{4}{5}\ln(x+3) + C \,.$$

---

9. Integrate $\int \frac{x^2dx}{\sqrt{9-x^2}}$.

Answer. We use the substitution indicated in figure 2. Then

$$x = 3\sin u, \;\; dx = 3\cos u\, du, \;\; \sqrt{9-x^2} = 3\cos u \,.$$

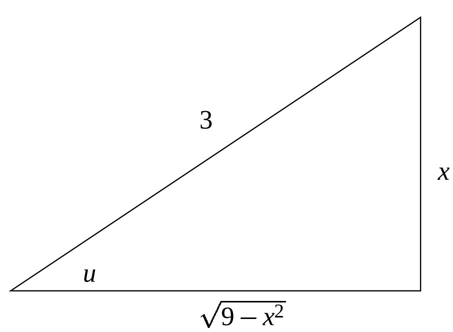

Figure 2

The integral becomes

$$\int \frac{xdx}{\sqrt{9-x^2}} = \int \frac{(9\sin^2 u)(3\cos u du)}{3\cos u} = 9\int \sin^2 u du \ .$$

This integral requires the half-angle formula; using that we obtain:

$$\int \sin^2 u du = \frac{1}{2}\int (1-\cos(2u))du = \frac{1}{2}(u - \frac{1}{2}\sin(2u)) + C \ ,$$

so

$$\int \frac{xdx}{\sqrt{9-x^2}} = \frac{9}{2}(u - \frac{1}{2}\sin(2u) + C \ .$$

Now, to express this in terms of x, we need the double angle formula $\sin(2u) = 2\sin u \cos u$, giving finally:

$$\int \frac{xdx}{\sqrt{9-x^2}} = \frac{9}{2}(\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}) + C \ .$$

---

10. Integrate $\int \frac{x^2dx}{\sqrt{9+x^2}}$.

Answer. We use the substitution indicated in figure 3.

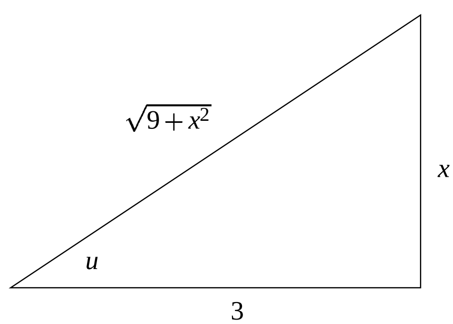

Figure 3

Then

$$x = 3\tan u \ , \ \ dx = 3\sec^2 u du, \ \sqrt{9+x^2} = 3\sec u \ ,$$

and the integral becomes

$$\int \frac{x^2dx}{\sqrt{9+x^2}} = \int \frac{(9\tan^2 u)(3\sec^2 u du)}{3\sec u} = 9\int \tan^2 u \sec^2 u du \ .$$

We calculate this integral by parts: $v = \tan u, \ dv = \sec^2 u du, \ dw = \sec u \tan u du, \ w = \sec u$:

$$\int \tan^2 u \sec^2 u du = \tan u \sec u - \int \sec^3 u du$$

$$= \tan u \sec u - \frac{1}{2}(\sec u \tan u + \ln|\sec u + \tan u|) + C = \frac{1}{2}(\sec u \tan u - \ln|\sec u + \tan u|) + C \ .$$

Resubstituting back, to get an expression in terms of x:

$$\int \frac{x^2dx}{\sqrt{9+x^2}} = \frac{9}{2}(\frac{\sqrt{9+x^2}}{3}\frac{x}{3} - \ln|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}|) + C \ , \quad \text{or}$$

$$\int \frac{x^2dx}{\sqrt{9+x^2}} = \frac{1}{2}x\sqrt{9+x^2} - \frac{9}{2}\ln|x + \sqrt{9+x^2}| + C \ .$$