Calculus II
Practice Problems 11: Answers

1. Show that the graph of the polar equation $r = a\cos\theta + b\sin\theta$ is a circle of radius $\sqrt{a^2+b^2}$ going through the origin. Where is its center?

Answer. Let $c = \sqrt{a^2+b^2}$ and $\theta_0 = \arctan(b/a)$, so that $a = c\cos\theta_0$ and $b = c\sin\theta_0$. Then the equation becomes

$$r = c(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0) = c\cos(\theta-\theta_0)$$

which is a circle through the origin with diameter c and center on the ray $\theta = \theta_0$. The center is at $(1/2)(c\cos\theta_0,\ c\sin\theta_0) = (a/2, b/2)$. This could also be seen by multiplying the given equation by r and changing to cartesian coordinates, getting

$$x^2 + y^2 = ax + by$$

and then completing the square.

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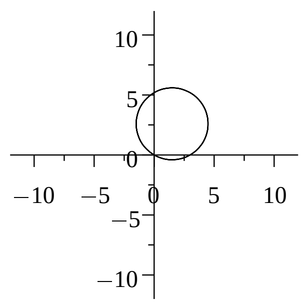
2. Graph $r = 3(\cos\theta + \sqrt{3}\sin\theta)$.

Answer. Following problem 1, we have $c = \sqrt{3^2 + (3\sqrt{3})^2} = 6$ and $\theta_0 = \arctan\sqrt{3} = \pi/3$. Thus, this is the circle $r = 6\cos(\theta - \pi/3)$, since $\arctan\sqrt{3} = \pi/3$. The center is at $(3/2, 3\sqrt{3}/2)$. For the graph, see the figure.



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3. What is the polar equation of an ellipse, with one focus at the origin, vertex at the point (-1,0) and directrix the line $x = -3$?
Answer. The general equation of such an ellipse is

$$r = \frac{ed}{1 - e\cos\theta}$$

where d is the distance between focus F and directrix L , and thus $d = 3$; and e is the eccentricity. Knowing the vertex tells us the eccentricity: from $|VF| = e|VL|$ we get $1 = e(2)$, so $e = 1/2$. Thus the equation is

$$r = \frac{3/2}{1 - (1/2)\cos\theta}\ .$$

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4. Identify the curve: $y = 2\sin(5\theta)$.

Answer. This is the five-petaled rose. The first petal lies between $\theta = 0$ and $\theta = \pi/5$ and has length 2.

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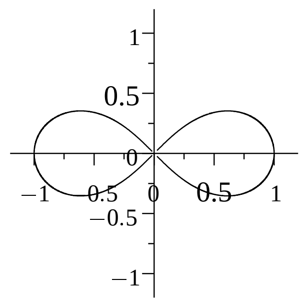
5. Graph $r^2 = \cos(2\theta)$. This is called a *lemniscate*.
Answer. We make this table of the values:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	1	0	$*$	0	1

with the star indicating that r is not defined when $\cos(2\theta) < 0$. But on the other hand, when $\cos(2\theta) > 0$, we should consider the negative square root as well, for r. Finally, as θ goes from π to 2π we get the image reflected in the x-axis. This gives the graph:



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6. Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.
Answer. Here $r = e^{2\theta},\ dr = 2e^{2\theta}d\theta$. Thus $ds^2 = 4e^{4\theta}d\theta^2 + e^{4\theta}d\theta^2$, and we have

$$Length \ = \ \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{5}e^{2\theta}d\theta = \frac{\sqrt{5}}{2}(e^{4\pi} - 1)$$

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7. Find the length of the spiral $r = e^{-\theta}$ for $\theta \geq 0$.

Answer. Here $r = e^{-\theta},\ dr = -e^{-\theta}d\theta$. Thus $ds^2 = e^{-2\theta}d\theta^2 + e^{-2\theta}d\theta^2 = 2e^{-2\theta}d\theta^2$, and we have

$$Length \ = \ \int_0^{\infty} ds = \sqrt{2}\int_0^{\infty} e^{-\theta}d\theta = \sqrt{2}\lim_{A\to\infty}(1 - e^{-A}) = \sqrt{2}$$

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8. Find the area inside the limaçon $r = 3 + \sin\theta$.

$$Area \ = \ \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi}(3 + \sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9 + 6\sin\theta + \sin^2\theta)d\theta =$$
$$\frac{1}{2}\int_0^{2\pi}(9 + \frac{1 - \cos(2\theta)}{2})d\theta = \frac{19}{2}\pi \ .$$

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9. Find the area inside the cardioid $r = 1 - \sin\theta$ and above the x-axis.

Answer. Since the cardioid is symmetric about the y-axis, the desired answer is twice the area inside the first quadrant:

$$Area \;=\; 2\int_0^{\pi/2}\frac{1}{2}r^2 d\theta = \int_0^{\pi/2}(1-\sin\theta)^2 d\theta = \int_0^{\pi/2}(1-2\sin\theta+\sin^2\theta)d\theta =$$

$$\int_0^{\pi/2}(\frac{3}{2}-2\sin\theta-\frac{\cos(2\theta)}{2})d\theta = \frac{3}{4}\pi-2$$

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10. What is the slope of the spiral $r=\theta$ at the points $\theta=2\pi n$ for n a positive integer? What about the spiral $r=e^{\theta}$ at the same points?

Answer. We use equation (15) of the notes. Since $r=\theta,\ dr/d\theta=1$, and we have, for the slope m (of the tangent line) of the first spiral:

$$m=\frac{\theta\cos\theta+\sin\theta}{-\theta\sin\theta+\cos\theta}\ ,$$

so at $\theta=2\pi n$, we get $m=2\pi n$. As for the logarithmic spiral, we saw, in example 26, that the spiral $r=e^{a\theta}$ makes a constant angle (whose tangent is a) with the ray from the origin. At the points $2\pi n$, this ray is the x-axis, and for our curve $a=1$; thus we have $m=1$ at all points. A calculation using (15) corroborates this:

$$m=\frac{e^{\theta}(-\sin\theta+\cos\theta)}{e^{\theta}(\cos\theta+\sin\theta)}=\tan(\theta-\pi/4)\ .$$