Calculus II
Practice Problems 10: Answers

In problems 1-4 put the conic in standard form, and find the center, vertices, foci.

1. $y - 8x^2 + 32x - 29 = 0$

Answer. Complete the square:

$$y - 8(x^2 - 4x + 4) - 29 + 32 = 0$$
$$y + 3 = 8(x-2)^2$$

This is a parabola with vertex at (2,-3) and axis the line $x = 2$. Since $4p = 8$, we have $p = 2$, so the focus is at $(2, -1)$.

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2. $9x^2 + 4y^2 - 36x + 8y + 4 = 0$

Answer. Complete the squares:

$$9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) + 4 - 36 - 4 = 0$$

leading to

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} = 1 \ .$$

This is an ellipse centered at $(2, -1)$, with major axis the line $x = 2$. Since $b = 3$, and $a = 2$, the vertices are at $(2, -1 \pm 3)$ or (2,2) and (2,-4). We have $c^2 = b^2 - a^2 = 5$, so the foci are at $(2, -1 \pm \sqrt{5})$.

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3. $4x^2 - y^2 + 2y = 5$

Answer. Complete the squares:

$$4x^2 - (y^2 - 2y + 1) = 5 - 1$$

leading to

$$x^2 - \frac{(y-1)^2}{4} = 1 \ .$$

This is a hyperbola centered at $(0, 1)$, with major axis the line $y = 1$. Since $a = 1$ and $b = 2$, the vertices are at $(\pm 1, 1)$. We have $c^2 = a^2 + b^2 = 1 + (1/4) = 5/4$, so the foci are at $(\pm\sqrt{5}, 1)$.

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4. $x^2 - 5y^2 - 4x + 10y = 1$

Answer. Complete the squares:

$$(x^2 - 4x + 4) - 5(y^2 - 2y + 1) = 1 + 4 - 5 = 0$$

leading to

$$(x-2)^2 = 5(y-1)^2$$

The graph is the pair of lines intersecting at (2,1): $x - 2 = \pm\sqrt{5}(y-1)$.

In problems 5-7, find the equation of the tangent line to the curve at the point (x_0, y_0) on the curve.

5. $x^2 - 5y = 0$, $(10, 20)$

Answer. We recall from example 11, Chapter I.5, how to find tangent lines by implicit differentiation. Taking differentials we have

$$2xdx - 5dy = 0$$

Now replace x, y by the coordinates of the point: 10, 20, and dx and dy by the increments along the tangent line. This gives us

$$2(10)(x-10) - 5(y-(20) = 0 \quad \text{or} \quad 20x - 5y = 100 \ .$$

6. $x^2 + 4y^2 = 16$, $(2\sqrt{3}, 1)$

Answer. Take differentials

$$2xdx + 8ydy = 0$$

and evaluate at the given point:

$$2(2\sqrt{3})(x - 2\sqrt{3}) + 8(1)(y-1) = 0 \quad \text{or} \quad (4\sqrt{3})x + 8y = 32$$

7. $4x^2 - y^2 = 1$, $(\sqrt{2}/2, 1)$

Answer. Take differentials

$$8xdx - 2ydy = 0 \quad \text{or} \quad dy = 4xdx$$

and evaluate at the given point:

$$y - 1 = 4(\sqrt{2}/2)(x - \sqrt{2}/2) \quad \text{or} \quad y = (2\sqrt{2})x - 1$$

In each of problems 8 and 9, the curve described depends upon a parameter. Identify the parameter, and find the equation of the curve in terms of the parameter.

8. A parabola with axis the x-axis and focus at the origin.

Answer. The equation of a parabola with axis the x-axis and vertex at $(x_0, 0)$ is

$$y^2 = 4p(x - x_0)$$

where p is the separation between the focus and the vertex. Since the focus is at the origin, the vertex is at $(p, 0)$, thus the desired equation is

$$y^2 = 4p(x - p)$$

9. A hyperbola with foci at (-1,0), (1,0).

Answer. Since the foci are on the x-axis and the origin is midway between the foci, this hyperbola has as its axis the x-axis, and its center is the origin. Place the vertices at the points $(\pm a, 0)$, with $a > 1$. Then, since $b^2 = a^2 - c^2 = a^2 - 1$, the desired equation is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2 - 1} = 1$$

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10. Find the point (x, y) on the parabola $y^2 = 12x$ for which the line from the focus meets the tangent line at an angle of 45°.

Answer. By the optical property of the parabola, the tangent line at (x, y) makes an angle of $45°$ with the horizontal, so the slope of the tangent line is $m = \tan 45° = 1$. Differentiating the equation of the curve, we have

$$2y\frac{dy}{dx} = 12 \quad \text{so that} \quad m = \frac{dy}{dx} = \frac{6}{y} .$$

Thus $6/y = 1$, so $y = 6$ and $x = 3$.