Calculus I
Practice Problems 9: Answers

1. Find the indefinite integral of:

a) $f(x) = (x^2+1)^2 x$
Answer. Let $u = x^2+1,\ du = 2xdx$ so that

$$\int (x^2+1)^2 xdx = \int \frac{1}{2}u^2 du = \frac{1}{2}\frac{1}{3}u^3 + C = \frac{1}{6}(x^2+1)^3 + C$$

b) $g(x) = (x^2-1)(x^3-3x)^3$
Answer. Let $u = x^3-3x,\ du = 3(x^2-1)dx$ so that

$$\int (x^2-1)(x^3-3x)^3 dx = \int \frac{1}{3}u^3 du = \frac{1}{3}\frac{1}{4}u^4 + C = \frac{1}{12}(x^3-3x)^4 + C$$

c) $h(x) = x(x^2-1)^{-3}$
Answer. Let $u = x^2-1,\ du = 2xdx$ so that

$$\int \frac{x}{(x^2-1)^3}dx = \int \frac{1}{2}u^{-3}du = \frac{1}{2}\frac{1}{(-2)}u^{-2} + C = -\frac{1}{4}(x^2-1)^{-2} + C$$

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2. Find the indefinite integral of:

a) $h(x) = \tan x \sec^2 x$
Answer. Le $u = \tan x,\ du = \sec^2 x dx$, so that

$$\int \tan x \sec^2 x dx = \int u du = \frac{1}{2}\tan^2 x + C\ .$$

b) $g(x) = \sin^3 x$
Answer. Here we first have to use the trigonometric identity: $\sin^2 x = 1 - \cos^2 x$ to write

$$\int \sin^3 x dx = \int \sin x(1-\cos^2 x)dx = \int \sin x dx - \int \cos^2 x \sin x dx\ .$$

Now, the first integral is in our tables, and for the second we make the substitution $u = \cos x,\ du = -\sin x dx$:

$$\int \sin^3 x dx = -\cos x + \int u^2 du = -\cos x + \frac{\cos^3 x}{3} + C\ .$$

c) $f(x) = \sin(2x)(\cos(2x))^2$
Answer. Let $u = \cos(2x),\ du = -2\sin(2x)dx$:

$$\int \sin(2x)(\cos(2x))^2 dx = -\frac{1}{2}\int u^2 du = -\frac{1}{6}u^3 + C = -\frac{1}{6}(\cos(2x))^3 + C\ .$$

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3. $\int x(x^2+1)^{-2}dx =$

Answer. Substitute $u = x^2+1,\ du = 2xdx$:

$$= \frac{1}{2}\int u^{-2}du = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C$$

4 a) Find the indefinite integral $\int \sqrt{x}(x+1)dx =$
Answer. First do the multiplication, and then integrate:

$$\int x^{1/2}(x+1)dx = \int (x^{3/2}+x^{1/2})dx = \frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C$$

b) Find the indefinite integral $\int x\sqrt{x+1}dx =$
Answer. Here we can't multiply through, but after the substitution $u = x+1$, we can. For then $x = u-1,\ du = dx$, so

$$\int x\sqrt{x+1}dx = \int (u-1)\sqrt{u}du = \int (u^{3/2}-u^{1/2})du = \frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+C$$

$$= \frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C$$

5. Find the solution to the differential equation $y' = y^2x^2+y^2$ such that $y(1) = 2$.

Answer. By separating the variables we write this as an equation of differentials:

$$y^{-2}dy = (x^2+1)dx$$

Now we integrate each side:

$$-y^{-1} = \frac{x^3}{3}+x+C\ ,$$

and solve for C using the initial condition $x = 1,\ y = 2$:

$$-\frac{1}{2} = \frac{1}{3}+1+C$$

so $C = -11/6$ and

$$y = \frac{1}{11/6-\frac{x^3}{3}-x}$$

6. Given

$$\frac{dy}{dx} = x^2\sqrt{y}, \quad y = 4 \quad \text{when} \quad x = 0$$

find y as a function of x.
Answer. Rewrite as

$$y^{-1/2}dy = x^2dx$$

$$2y^{1/2} = \frac{x^3}{3}+C$$

Using the initial condition $x = 0,\ y = 4:\ 2(4)^{1/2} = 0^3/3+C$, so $C = 4$, and

$$2y^{1/2} = \frac{x^3}{3}+4$$

which resolves to

$$y = (\frac{x^3}{6}+2)^2$$

7. Find y as a function of x, given

$$\frac{dy}{dx} = \frac{\sin x}{y}\ , \qquad y = 5 \quad \text{when} \quad x = 0\ .$$

Answer. Rewrite as $ydy = \sin x dx$, and integrate; $y^2/2 = -\cos x + C$. Use the initial condition to find $C:\ 25/2 = -1 + C$, so $C = 27/2$. Thus $y^2 = 27 - 2\cos x$, or $y = \pm\sqrt{27 - 2\cos x}$.

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8. Find $f(x)$ given that $f(2) = 1,\ f'(1) = -1$ and $f''(x) = x - x^{-3}$.

Answer. Integrating,

$$f'(x) = \frac{x^2}{2} + \frac{1}{2}x^{-2} + C$$

Using $f'(1) = -1$ we get $C = -2$. Put in this value of C and integrate again, getting

$$f(x) = \frac{x^3}{6} - \frac{1}{2}x^{-1} - 2x + C$$

Using $f(2) = 1$ we get $C = 47/12$ so

$$f(x) = \frac{x^3}{6} - \frac{1}{2}x^{-1} - 2x + \frac{47}{12}$$

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9. An automobile is travelling down the road a speed of 100 ft/sec. a) At what constant rate must the automobile decelerate in order to stop in 300 ft.? b) How long does that take?

Answer. Let a be the acceleration to find. The equations of motion are

$$s = -\frac{a}{2}t^2 + v_0 t + s_0\ , \quad v = -at + v_0\ .$$

At the beginning of the deceleration we have $s_0 = 0,\ v_0 = 100$, and at the end $s = 300,\ v = 0$. Thus our equations are (with t now representing the time to stop):

$$300 = -\frac{a}{2}t^2 + 100t, \quad 0 = -at + 100\ .$$

From the second, we have $t = 100/a$, putting that in the first equation we obtain

$$300 = -\frac{a}{2}\frac{10^4}{a^2} + 10^2\frac{10^2}{a}\ ,$$

from which we obtain $300 = 10^4/(2a)$, so that $a = 10^2/6 = 16.67$ ft/sec^2, or slightly more than half the acceleration due to gravity. Finally, $t = 10^2/a = 6$ seconds.

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10. A ball is thrown from ground level so as to just reach the top of a building 150 ft. high. At what initial velocity must the ball be thrown?

Answer. Following the equations of motion, we have $dv/dt = -32$ ft/sec^2, so $v = -32t + v_0$ and $s = -16t^2 + v_0 t$, taking the initial conditions $s_0 = 0$ and v_0 is the initial velocity to be found. Our conditions are that $v = 0$ when $s = 150$, so we have to solve the pair of equations

$$0 = -32t + v_0\ , \quad 150 = 16t^2 + v_0 t\ .$$

Write t in terms of v_0 using the first equation; substitute that expression in the second and solve for v_0. Alternatively we can use the conservation of energy as expressed in equation (4.45):

$$-\frac{1}{2}v_0^2 = -32(150) \ ,$$

from which we get $v_0^2 = 64\sqrt{150}$, or $v_0 = 8\sqrt{150} = 97.98$ ft/sec.