Calculus I
Practice Problems 7: Answers

1. Find the dimensions of the right triangle with one vertex at the origin, the right angle on the positive x-axis, and the third vertex on the curve $y = 4 + x^{-2}$ which is of minimum area.

Answer. First, we draw the diagram (see below). The variables are: area A, the length of the base x, and the height y of the triangle. The variables are related by

$$A = \frac{1}{2}xy \ , \qquad y = 4 + x^{-2} \ ,$$

so $A = (1/2)x(4 + x^{-2}) = 2x + (1/2)x^{-1}$. Differentiating:

$$\frac{dA}{dx} = 2 - \frac{1}{2x^2} \ .$$

Setting this equal to zero we get $x = 1/2$. Checking that the sign of dA/dx changes from negative to positive as we pass $x = 1/2$, we find that the minimum occurs when $x = 1/2$, and $y = 8$, so the minimum area is $A = 2$.

---

2. We are asked to make an open-topped box out of a rectangular sheet of tin 24 in. wide and 45. in long. This is to be done by cutting congruent squares out of each corner of the sheet and then bending sides upward to from the sides of the box. What are the dimensions of the box of greatest volume?

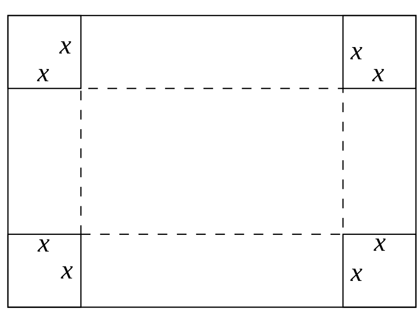
Answer. In the corresponding figure x represents the side of the cut-out square, which is also the height. The other dimensions are $24 - 2x,\ 45 - 2x$, so the volume of the resulting box is

$$V = x(25 - 2x)(45 - 2x) = 4x^3 - 138x^2 + 1080x \ .$$

Differentiate and set equal to zero, obtaining the equation

$$12(x^2 - 23x + 90) = 0 \ .$$

The roots are 5 and 18; the latter doesn't fit the context, so the answer is $x = 5$. Thus the dimensions of the box should be 5,14,35, and the maximum volume is $V = 2450$ in^2.



---

3. An object is moving on the plane. It starts at one foot up the y-axis (at $(0,1)$), and travels to a point 3 ft down the x-axis (at $(3,0)$) by heading straight for some point $(x,0)$ at 2 ft/sec, and then along the x-axis at 3 ft/sec. Find the value of x which minimizes the time it takes.

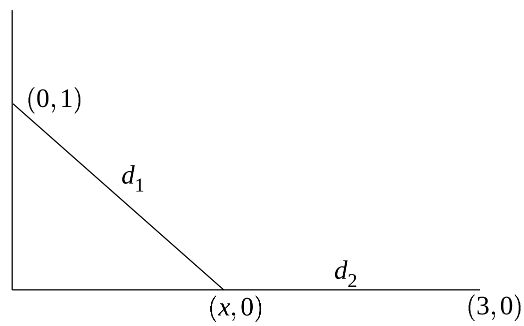
Answer. The path of travel of the particle is shown in the figure. The distance traveled at 2 ft/sec is $\sqrt{1+x^2}$, and distance travelled at 3 ft/sec is $3-x$. Using (rate)(time) = distance, we see that the total time taken is

$$T = \frac{\sqrt{1+x^2}}{2} + \frac{3-x}{3} \; .$$

To minimize T, we differentiate:

$$\frac{dT}{dx} = \frac{1}{2}\frac{x}{\sqrt{1+x^2}} - \frac{1}{3} \; .$$

We set this equal to zero and solve, obtaining the equation $3x = 2\sqrt{1+x^2}$. Squaring both sides and solving for x leads to $5x^2 = 4$, so $x = 2/\sqrt{5}$.



4. An automobile maufacturer sells, on average, 8000 cars per month at \$25 thousand dollars. The marketing department has determined that for every one thousand dollar reduction in price, the company can sell an additional 500 cars per month. At what price should the car be sold so as to maximize revenue?

Answer. Let p be the price reduction (in thousands of dollars) at which the automobile is to be sold. With this reduction, the manufacturer sells $8+p/2$ thousand cars, and the price is $25-p$. Thus the total revenue is $R = (8+p/2)(25-p)$. To maximize revenue we differentiate, and set the derivative equal to zero:

$$R' = \frac{1}{2}(25-p) - (8+p/2) = 4.5 - p \; .$$

The desired reduction is $p =$ \$4500 and the selling price should be \$ 20,500.

5. The temperature at a point x feet from a heating source is proportional to I/x^2 where I is the intensity of the source. Suppose that two heaters, one three times as intense as the other, are place 60 feet apart. At what point between the heaters is the temperature a minimum?

Answer. At a point x feet away from the weaker source the amount of heat produced by the two heaters is proportional to

$$y = \frac{1}{x^2} + \frac{3}{(60-x)^2} \; .$$

We want to minimize y, so we calculate the derivative:

$$y' = -2x^{-3} + 6(60-x)^{-3} \; .$$

Set this equal to zero and solve for x:

$$\frac{1}{(60-x)^3} = \frac{1}{3x^3} \quad \text{or} \quad \left(\frac{x}{60-x}\right)^3 = \frac{1}{3} \; .$$

Take the cube root of both sides to obtain

$$\frac{x}{60-x} = \frac{1}{3^{1/3}} \quad \text{which has the solution} \quad x = \frac{60}{1+3^{1/3}} \, .$$

---

6. A rectangular racecourse is to be made so the diagonal measures 5 furlongs, and so we can place 20 spectators per furlong along the horizontal sides, and 30 spectators per furlong along the vertical sides. What should the dimensions of the course be so that number of spectators is the maximum?

Answer. Let x be the length of the horizontal side of the the racecourse, y the length of the vertical side, and S the number of spectators. Since the diagonal is to be kept at 5, we have the constraint $x^2+y^2=25$. For a given x and y, we can sit $S=40x+60y$ spectators around the course. Then, in terms of x alone, $S=40x+60\sqrt{25-x^2}$, and x ranges between 0 and 5. Differentiate:

$$S' = 40 - \frac{60x}{\sqrt{25-x^2}}$$

$S'=0$ when $40\sqrt{25-x^2}=60x$, or

$$\sqrt{25-x^2} = \frac{3}{2}x \, .$$

Squaring both sides and solving for x, we find $x=\sqrt{100/13}=2.78$ and then $y=4.16$. The points to be considered are thus $x=0,2.78,5$, and the corresponding values of S are 300,360,200, so at $x=2.78$, $y=4.16$ we have the maximum number of spectators.

---

7. Farmer Brown wants to build a right triangular chicken coop containing 100 square feet. The hypotenuse will lie on an existing wall, but the other two sides are to be built. What should the dimensions of these sides be so as to minimize the sum of their lengths?

Answer. Letting x and y be the length of the sides of the triangle we obtain the constraint equation $(1/2)xy=100$ and the objective function (the total length) is $L=x+y$. Then $L=x+200x^{-1}$. The derivative is $L'=1-200x^{-2}$, which has the root $x=10\sqrt{2}$, and also $y=10\sqrt{2}$. We should have observed that since the equations are symmetric in x and y, that the solution must have $x=y$.

---

8. The Jones Jumpersuit Company can sell $400-8p$ jumpersuits each month at a price of $120+p$ dollars. Jones has fixed costs of \$ 8000 per month, and the cost in labor and material for each suit is \$ 25. At what price will Jones maximize profit?

Answer. The cost of producing $400-8p$ jumpersuits is

$$Cost = 8000+25(400-8p) = 18000-200p$$

and the income from this sale is

$$Income = (400-8p)(120+p) = 48000-560p-8p^2 \, .$$

Profit is

$$Profit \;=\; Income \;-\; Cost \;= 48000-560p-8p^2-(18000-200p) = -8p^2-360p+30000$$

Differentiate and set equal to zero: $R' = -16p-360$, so we shoud take $p=\$-22.50$. This is the value of p which maximizes profit, so the price to maximize profit is $120+p=\$97.50$.