Math 1210-90 Calculus I
Examination 3 Answers

1. Find the function $y = f(x)$ which satisfies the differential equation $x^2y' + (1+x^2)y^2 = 0$ such that $f(1) = 2$.

Solution. First, separate the variables:

$$\frac{dy}{y^2} = -\frac{(1+x^2)dx}{x^2} = -(x^{-2}+1)dx \ .$$

Now, integrate both sides:

$$-\frac{1}{y} = \frac{1}{x} - x + C \ .$$

Put in the initial conditions $x = 1, y = 2$ to solve for C:

$$-\frac{1}{1} = \frac{1}{1} - 1 + C \quad \text{so} \quad C = -\frac{1}{2} \ .$$

Thus

$$\frac{1}{y} = x - \frac{1}{x} + \frac{1}{2} \quad \text{or} \quad y = \frac{1}{x - \frac{1}{x} + \frac{1}{2}} \ .$$

2. Find the definite Integrals:

a) $$\int_{-\pi/2}^{\pi/2} \cos^2 x \sin x dx =$$

Solution. Let $u = \cos x, \ du = -\sin x dx$. Then

$$\int_{-\pi/2}^{\pi/2} \cos^2 x \sin x dx = -\int_{-1}^{1} u^2 du = -\frac{u^3}{3}\Big|_{-1}^{1} = -\frac{1}{3} - (-\frac{1}{3}) = 0 \ .$$

One could also observe that the interval is symmetric about the origin and the integrand is an odd function, so the integral is 0.

b) $$\int_1^2 (x^2 + x^{-2})dx =$$

Solution.

$$\int_1^2 (x^2 + x^{-2})dx = \frac{x^3}{3} - \frac{1}{x}\Big|_1^2 = \frac{8}{3} - \frac{1}{2} - (\frac{1}{3} - 1) = \frac{7}{3} + \frac{1}{2} = \frac{17}{6} \ .$$

3. Find the area of the region above the x-axis and below the curve $y = \sec^2 x$ lying between the lines $x = -\pi/4$ and $x = \pi/4$,

Solution. This is the integral

$$\int_{-\pi/4}^{\pi/4} \sec^2 x dx = \tan x \Big|_{-pi/4}^{\pi/4} = 1 - (-1) = 2 \ .$$

4. The region in the first quadrant between the coordinate axes and the curve $y = 1 - x^{2/3}$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution. Sweep the volume out in the y direction, usng the method of discs. Then $dV = \pi x^2 dy$, and (solving for x in terms of y), $x = (1-y)^{3/2}$. The volume then is

$$\pi \int_0^1 (1-y)dy = -\pi \int_1^0 udu = -\pi \frac{u^2}{2}\Big|_1^0 = \frac{\pi}{2} \ .$$

If instead, we sweep out in the x direction, we must use the shell method. Here $dV = 2\pi x(1 - x^{2/3})dx = 2\pi(x - x^{5/3})dx$, and the volume is

$$\int_0^1 (x - x^{5/3})dx = 2\pi(\frac{x^2}{2} - \frac{3}{8}x^{8/3})\Big|_0^1 = 2\pi(\frac{1}{2} - \frac{3}{8}) = \frac{\pi}{2} \ .$$

5. The region bounded by the x-axis and the curve $y = 2x - x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution. Here we use the method of discs, integrating in the x direction. $dV = \pi(2x - x^2)^2 dx$, so the volume is

$$\pi \int_0^2 (2x - x^2)^2 dx = \pi \int_0^2 (4x^2 - 4x^3 + x^4)dx = \pi(\frac{4}{3}x^3 - x^4 + \frac{x^5}{5})\Big|_0^2 = 4\pi(\frac{4}{3} - 2 + \frac{8}{5}) = \frac{56\pi}{15} \ .$$