Math 1210-90 Calculus I
Examination 2, Oct 23,25, 2003, Answers
WARNING: You must show work, particularly where graphing is involved.

1. A curve in the plane is given implicitly by the equation

$$2x^2 + 2xy + 3y^2 = 40 \ .$$

At what points does the curve have a horizontal tangent line?

Solution. We differentiate implicitly by taking the differential of both sides:

$$4xdx + 2xdy + 2ydx + 6ydy = 0 \quad \text{which simplifies to} \quad (4x + 2y)dx + (2x + 6y)dy = 0 \ .$$

We see that we must have $dy/dx = 0$ (the condition for a horizontal tangent) when $4x + 2y = 0$, or $y = -2x$. Substituting that in the equation of the curve, we get

$$2x^2 - 4x^2 + 12x^2 = 40 \quad \text{or} \quad 10x^2 = 40 \ ,$$

so that $x = \pm 2$ and thus $y = \mp 4$. The points at which there is a horizontal tangent are thus (2,-4) and (-2,4).

If the issue were to find the points at which the tangent line is vertical, then we set the coefficient of dy equal to zero, obtaining $x = -3y$, with the solutions $\pm(-3\sqrt{8/3}, \sqrt{8/3})$.

2. A pool filled with water is shaped like a box lying over a rectangle of area 60 ft^2. Because of a break in the bottom, water begins to leak out of the pool at a rate proportional to the volume V of water in the pool according to the formula

$$\frac{dV}{dt} = \frac{1}{20}V \ .$$

V is measured in ft^3 and time in minutes. At what rate is the height of the water in the pool decreasing when the height is 6 feet? Remember that the volume of water is equal to the area of the base times the height of the water.

Solution. The relationship between volume and height is $V = 60h$, since the area of the base is 60 ft^2. Thus $60dh/dt = dV/dt$. Then,

$$\frac{dh}{dt} = \frac{1}{60}\frac{dV}{dt} = \frac{1}{60}\frac{1}{20}V = \frac{1}{60}\frac{1}{20}60h = \frac{1}{20}h = \frac{6}{20} = .3$$

feet/min when h is 6 feet.

3. What point on the line $2x + y = 10$ is closest to the origin?

Solution. The distance of (x, y) from the origin is $\sqrt{x^2 + y^2}$. Thus we want to minimize $x^2 + y^2$ subject to the condition $y = 10 - 2x$. We take $f(x) = x^2 + (10 - 2x)^2$; the minimum of f is to be found among the points at which $f'(x) = 0$. Now

$$f'(x) = 2x + 2(10 - 2x)(-2) = 10x - 40 = 0$$

which has the solution $x = 4$. At $x = 4$, $y = 10 - 2(4) = 10 - 8 = 2$, so the answer is $(4, 2)$.

4. What is the maximum of $$y = \frac{x}{x^2 + 1} \quad ?$$

Solution. We calculate the derivative:

$$\frac{dy}{dx} = \frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2} \ .$$

This is zero when $x^2 + 1 - x(2x) = 1 - x^2 = 0$, so at the points $x = \pm 1$. For $x = 1,\ y = 1/2$ and for $x = -1,\ y = -1/2$, thus the maximum value is $1/2$.

To be perfectly precise, we need to verify that the function has a maximum. But, since $|x| < x^2 + 1$ for all x, and $y \to 0$ as $|x| \to \infty$, we must have a point at which the maximum is achieved.

5. Graph $$y = \frac{x^2}{x^2 - 1}$$

showing clearly in what intervals the graph is increasing and decreasing.

Solution. First of all, the graph will have vertical asymptotes at $x = \pm 1$. To find out where it is increasing or decreasing, we calculate the derivative:

$$\frac{dy}{dx} = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} \ .$$

Thus $dy/dx > 0$ for $x < 0$ and $dy/dx < 0$ for $x > 0$. The curve is thus increasing in the left half plane, and decreasing in the right half plane.