1210-90 Final Exam

Summer 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers.

1. (16pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (4pts) $D_x(x^3 - 4x^2 + 9)$

4

$$= 3x^2 - 8x$$

   (b) (4pts) $D_x(\sqrt{x}\cos x)$ $= D_x\left(x^{1/2}\cos x\right)$

4

$$= \tfrac{1}{2}x^{-1/2}\cos x - x^{1/2}\sin x$$

$$= \frac{1}{2\sqrt{x}}\cos x - \sqrt{x}\sin x$$

   (c) (4pts) $D_x(\frac{\sin x}{x - \cos x})$

4

$$= \frac{(x-\cos x)(\cos x) - (\sin x)(1+\sin x)}{(x-\cos x)^2} = \frac{x\cos x - \sin x - 1}{(x-\cos x)^2}$$

   (d) (4pts) $D_x(\cos^7(x^2))$ **Note:** $\cos^7(x^2)$ is another way of writing $(\cos(x^2))^7$.

4

$$= 7\cos^6(x^2)(-\sin(x^2))(2x)$$

$$= -14x\cos^6(x^2)\sin(x^2)$$

2. (8pts) Compute the following limits. Answers may be $\pm\infty$ or 'DNE'. Show your work below and circle your final answer.

   (a) (4pts) $\lim\limits_{x\to\infty} \frac{x^2 - x}{4 - 5x^2}$ $= \lim\limits_{x\to\infty} \frac{x^2(1-\frac{1}{x})}{x^2(\frac{4}{x}-5)} = \lim\limits_{x\to\infty} \frac{1-\frac{1}{x}}{\frac{4}{x}-5} = \frac{1}{-5} = -\frac{1}{5}$

4

$$\lim_{x\to\infty} \frac{x^2-x}{4-5x^2} \approx \lim_{x\to\infty} \frac{x^2}{-5x^2} = -\frac{1}{5}$$

   (b) (4pts) $\lim\limits_{x\to 0} \frac{|x|}{x}$

4

when $x > 0$, $\frac{|x|}{x} = 1$, so $\lim\limits_{x\to 0^+} \frac{|x|}{x} = 1$.

when $x < 0$, $\frac{|x|}{x} = -1$, so $\lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$

not equal so $\lim\limits_{x\to 0} \frac{|x|}{x}$ DNE.

correct ans no work +2

24

3. (6pts) Use differentials to estimate the increase in the area of a circle when the radius increases from 4 cm to 4.25. Remember, the area of a circle is $A = \pi r^2$.

$A = \pi r^2$

6 $\quad dA = 2\pi r\, dr$ 3

$r = 4\text{cm}, \quad dr = .25 \text{ cm}$

So

$dA = 2\pi(4)(.25) = 2\pi \text{ cm}^2$

4. (6pts) Use implicit differentiation to find the slope of the tangent line to the ellipse at the point $(1, -3)$.

$$x^2 + \frac{y^2}{9} = 2$$

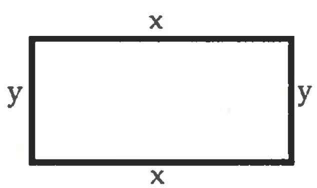
Diff. both sides w.r.t x:

6 $\quad 2x + \frac{2}{9} y \frac{dy}{dx} = 0$ 3

Plug in $x = 1, y = -3$

$2(1) + \frac{2}{9}(-3)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 3$

5. (10pts) Show that the rectangle of area 100 cm^2 with the smallest perimeter is the square with sides of length 10 cm. Do this by minimizing the perimeter $P = 2x + 2y$ assuming that the area $A = xy = 100$ cm^2. **Note:** You must use calculus to get credit!



$P = 2x + 2y$

$100 = xy \Rightarrow y = \frac{100}{x}.$

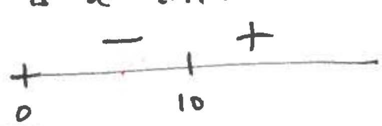
10 So $\quad P(x) = 2x + \frac{200}{x}.$ 4

Find critical pts:

$0 = P'(x) = 2 - \frac{200}{x^2} \Rightarrow 2x^2 = 200$

$\Rightarrow x = 10.$ 3

Check $x = 10$ is a minimum: 1



sign of P'

So $\quad x = y = 10$ cm 2

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6. (13pts) Consider the function

$$f(x) = \frac{x}{4+x^2}$$

(a) (4pts) Find $f'(x)$.

4 $f'(x) = \dfrac{(4+x^2)(1) - (x)(2x)}{(4+x^2)^2} = \dfrac{4-x^2}{(4+x^2)^2}$

(b) (4pts) Find the critical point(s) of $f(x)$.

4 $0 = \dfrac{4-x^2}{(4+x^2)^2} \Longrightarrow 0 = 4 - x^2 \Longrightarrow x = \pm 2$

2 (c) (2pts) Fill in the blanks: $f(x)$ is increasing on the interval ($\underline{-2}$, $\underline{2}$). **Note:** $\pm\infty$ are acceptable answers.

Sign of f': $-$ | -2 | $+$ | 2 | $-$

(d) (3pts) Which critical point is a local minimum?

3 $x = -2$ is a local min

answer: 1
Justification: 2

7. (6pts) Consider the function

$$f(x) = \sqrt[3]{x} + x = x^{1/3} + x$$

(a) (4pts) Find $f''(x)$.

4 $f'(x) = \frac{1}{3}x^{-2/3} + 1$

$f''(x) = \frac{-2}{9}x^{-5/3}$

(b) (2pts) Fill in the blanks: $f(x)$ is concave up on the interval ($\underline{-\infty}$, $\underline{0}$). **Note:** $\pm\infty$ are acceptable answers.

2 $x = 0$ is inflection pt:

$+$ | 0 | $-$ sign f''

8. (6pts) Evaluate the Riemann sum for $f(x) = 4 - x - x^2$ on the interval $[-1, 5]$ using the partition of 3 subintervals of equal length and the sample points being the left-endpoints of each subinterval.

-1 0 1 2 3 4 5

$\Delta x = 2$ +1

6 $\Delta x\left(f(-1) + f(1) + f(3)\right) = 2(4 + 2 + -8)$

$= -4$

+3 +1pnt +1

25

9. (12pts) Suppose $F(t)$ is an antiderivative of the function $f(t) = \cos(\sqrt{t})$.

2 (a) (2pts) Then $F'(t) =$ $\underline{\cos(\sqrt{t})}$.

(b) (2pts) According to the Second Fundamental Theorem of Calculus,

2
$$\int_1^x \cos(\sqrt{t})\, dt = F(\underline{\ x\ }) - F(\underline{\ 1\ }).$$

(c) (3pts) Use the above or the First Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\int_1^x \cos(\sqrt{t})\, dt$$

3

$$= \cos(\sqrt{x})$$

(d) (5pts) Find

$$\frac{d}{dx}\int_1^{x^4} \cos(\sqrt{t})\, dt$$

5

$$= \frac{d}{dx}\left(F(x^4) - F(1)\right) = F'(x^4)\cdot 4x^3$$

$$= \cos(x^2)\cdot 4x^3 \quad +2$$

10. (15pts) Find the following antiderivatives. **Remember:** $+C$!

Missing + C -2 total

(a) (5pts) $\int (x - x^3 + 2)\, dx$

5

$$\frac{x^2}{2} - \frac{x^4}{4} + 2x + C$$

(b) (5pts) $\int x(\frac{1}{2}x^2 + 9)^6\, dx$

$u = \frac{1}{2}x^2 + 9$

$du = x\, dx$

5

$$= \int u^6\, du = \frac{1}{7}u^7 + C$$

$$= \frac{1}{7}\left(\frac{1}{2}x^2 + 9\right)^7 + C$$

(c) (5pts) $\int \cos x\sqrt{1 + \sin x}\, dx$

$u = 1 + \sin x$

$du = \cos x\, dx$

5

$$= \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1 + \sin x)^{3/2} + C$$

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11. (26pts) Consider the region R in the first quadrant bounded by $y = 4x$ and $y = x^2$. Figure A below is a rough sketch of the region R.

(a) (5pts) Find the coordinates (x, y) of the point of intersection of the curves $y = 4x$ and $y = x^2$ labeled P in the sketch below.

$$4x = x^2 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0$$

$$x = 0, 4$$

$$(4, 16)$$

5

(b) (5pts) Find the area of the region R.

$$A = \int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{x^3}{3}\right|_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$$

5

(c) (8pts) Find the volume of the solid obtained by rotating the region R around the x-axis.

$$V = \int_0^4 \pi\left((4x)^2 - (x^2)^2\right)dx = \pi\int_0^4 (16x^2 - x^4)\,dx$$

$$= \pi\left(\frac{16}{3}x^3 - \frac{x^5}{5}\right|_0^4$$

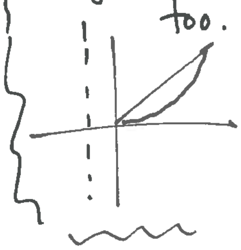
$$= \pi\left(\frac{16}{3}(64) - \frac{1}{5}(4^5)\right) = \frac{2048\pi}{15}$$

8

(d) (8pts) Find the volume of the solid obtained by rotating the region R around the line $y = -1$.

$$V = \int_0^4 \pi\left((4x+1)^2 - (x^2+1)^2\right)dx = \pi\int_0^4 (14x^2 + 8x - x^4)\,dx$$

I had meant to ask $x = -1$ ... this is a good problem too.



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$$= \pi\left(\frac{14}{3}x^3 + 4x^2 - \frac{x^5}{5}\right|_0^4$$

$$= \pi\left(\frac{14}{3}(64) + 64 - \frac{4^5}{5}\right)$$

$$= \frac{2368\pi}{15}$$

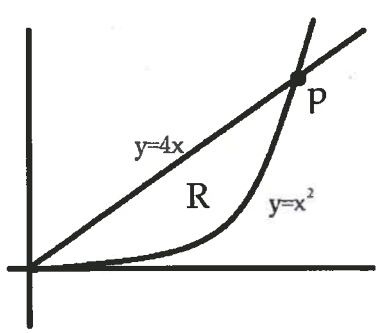

Figure A

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12. (8pts) Find the arc length of the curve defined by the parametric equations

$$x = \frac{1}{2}t^2 \qquad y = \frac{1}{3}(2t+1)^{3/2}$$

for $0 \le t \le 2$.

$x'(t) = t \qquad y'(t) = \frac{1}{2}(2t+1)^{1/2}\cdot 2 = (2t+1)^{1/2}$

8

$$L = \int_0^2 \sqrt{(t)^2 + ((2t+1)^{1/2})^2}\,dt = \int_0^2 \sqrt{t^2+2t+1} = \int_0^2 (t+1)\,dt$$

$$= \left(\frac{t^2}{2} + t \right|_0^2 = 4$$

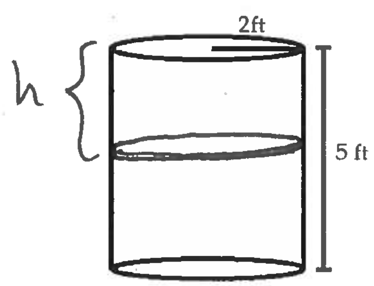
13. (6pts) A 1 meter-long spring requires a force of 5 Newtons to compress it 5 centimeters. How much work is required to compress the spring from it natural length to a length of .8 meters? **Remember:** springs satisfy Hooke's Law: $F = kx$, where k is the spring constant.

$5N = k(.05\,m) \Rightarrow k = 100.$

6

$$W = \int_0^{.2} 100x\,dx = \left(50x^2\right|_0^{.2} = 50(.2)^2 = 2\,J$$

14. (12pts) A cylindrical tank is filled with water. The tank is 5 feet tall and has a radius of 2 feet at its rim. Use an integral to determine much work is required to pump the water over the top edge of the tank? To simplify your calculations, use the symbol ρ to denote the density of water in lbs/ft^3.



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Volume of slab at location h:

$\pi(2)^2 dh = 4\pi\,dh\ ft^3$

Weight of slab at location h:

$4\rho\pi\,dh$ lbs

Distance lifted : h ft.

$$W = \int_0^5 4\rho\pi h\,dh = 4\rho\pi \int_0^5 h\,dh = 2\rho\pi \left(h^2\right|_0^5$$

$$= 50\rho\pi\ lb\cdot ft$$

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