1210-90 Final Exam
Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all work in the space provided. Please circle your final answer.

1. (20pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (5pts) $D_x(4x^3 - x^2 + 1)$

   $12x^2 - 2x$

   (b) (5pts) $D_x(x^3 \sin x)$

   $3x^2 \sin x + x^3 \cos x$

   (c) (5pts) $D_x(\frac{\cos x}{x})$

   $\frac{-x \sin x - \cos x}{x^2}$

   (d) (5pts) $D_x(\sin(x^{10}))$

   $\cos(x^{10})(10x^9)$

2. (10pts) Compute the following limits. Show your work below and circle your final answer.

   (a) (5pts) $\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+1) = 3$

   (b) (5pts) $\lim_{h\to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h}$ = derivative of $f(x) = \cos x$ at $x = \pi$.
   Hint: Think derivative!

   $f'(x) = -\sin x$

   $f'(\pi) = -\sin \pi = 0$

3. (10pts) Find the equation of the tangent line to the curve determined by

$$x^2 + xy^3 = y + 5$$

at the point $(2, 1)$.

Dif. both sides

$2x + y^3 + 3xy^2y' = y'$ 4

Plug in (2,1)

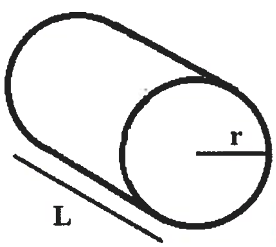
$4 + 1 + 6y' = y' \Rightarrow y' = -1.$ 3

$y - 1 = -1(x - 2)$

$y = -x + 3$

3

4. (12pts) A manufacturer wants to make a closed cylindrical tube out of 6π ft^2 of cardboard. The tube has length L feet and the radius of the end caps is r feet (see picture below). Find the values of L and r such that volume enclosed is at a maximum. **Note:** You must use calculus to get full credit!



$V = \pi r^2 L$ 3

maximize, with constraint

$6\pi = SA = 2\pi r L + 2\pi r^2$ 3

So $2\pi r L = 6\pi - 2\pi r^2$

$L = \frac{3}{r} - r$

So $V(r) = \pi r^2\left(\frac{3}{r} - r\right) = \pi(3r - r^3)$ 3

Find critical pts

$0 = V'(r) = \pi(3 - 3r^2) \Rightarrow r = 1.$ 3

Note $V''(1) = -6 < 0 \Rightarrow$ maximum.

when $r = 1$, $L = \frac{3}{1} - 1 = 2.$

$r = 1$
$L = 2$

5. (18pts) Consider the function

$$f(x) = x^4 - 6x^2 - 3$$

(a) (3pts) Find $f'(x)$.

$f'(x) = 4x^3 - 12x$

(b) (3pts) Find the critical point(s) of f.

$0 = 4x^3 - 12x = 4x(x^2-3)$

$x = 0, \pm\sqrt{3}$

(c) (4pts) Fill in the blanks: $f(x)$ is decreasing on the two intervals ($-\infty$, $-\sqrt{3}$) and (0 , $\sqrt{3}$).
Note: $\pm\infty$ are acceptable answers.

sign of f': $-$ | $+$ | $-$ | $+$ at $-\sqrt{3}$, 0, $\sqrt{3}$

(d) (3pts) ~~Find $f''(x)$.~~

$f''(x) = 12x^2 - 12$

(e) (3pts) Find the inflection point(s) of f.

$0 = 12x^2 - 12 \Rightarrow x = \pm 1$

(f) (2pts) Fill in the blanks: $f(x)$ is concave down on the interval (-1 , 1). **Note:** $\pm\infty$ are acceptable answers.

sign of f'': $+$ | $-$ | $+$ at -1, 1

6. (8pts) Evaluate the Riemann sum for $f(x) = \frac{x^2+15}{x}$ on the interval $[0, 6]$ using the partition of 3 subintervals of equal length with the sample points being the midpoints of each subinterval.

0 1 2 3 4 5 6

$\Delta x = 2.$

$f(1)\Delta x + f(3)\Delta x + f(5)\Delta x = (16)(2) + (8)(2) + (8)(2)$

$= 32 + 16 + 16$

$= 64$

7. (15pts) Use both versions of the Fundamental Theorem of Calculus to evaluate the following:

(a) (5pts) $\int_0^1 (x^2 - 1)\, dx$

$= \left(\frac{x^3}{3} - x \right|_0^1 = \left(\frac{1}{3} - 1\right) - (0 - 0) = -\frac{2}{3}$

(b) (5pts) $\frac{d}{dx} \int_1^x \sin(t^3)\, dt$ $= \sin(x^3)$

Note: You **do not** need to know an antiderivative of $\sin(t^3)$ to answer this question or (c) below.

(c) (5pts) $\frac{d}{dx} \int_1^{x^2} \sin(t^3)\, dt$. Let F be an antiderivative of $\sin(x^3)$

$\frac{d}{dx} \int_1^{x^2} \sin(t^3)\, dt = \frac{d}{dx}\left(F(x^2) - F(1)\right) = F'(x^2)(2x)$

$= \sin(x^6)(2x)$

8. (15pts) Find the following antiderivatives. **Remember:** $+C$!

(a) (5pts) $\int (x^2 - x + 1)\, dx$

$\frac{x^3}{3} - \frac{x^2}{2} + x + C$

(b) (5pts) $\int x \cos(x^2)\, dx$ $= \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin(u) + C$

$u = x^2$

$du = 2x\, dx$

$= \frac{1}{2} \sin(x^2) + C$

(c) (5pts) $\int \sin^2(3x) \cos(3x)\, dx$ $= \frac{1}{3} \int u^2 du = \frac{1}{9} u^3 + C$

$u = \sin(3x)$

$du = 3\cos(3x)$

$= \frac{1}{9} \sin^3(3x) + C$

9. (16pts) Consider the region R bounded by $y = 3 - x$, $x = 1$, $x = 2$ and the x-axis. Figure A below is a rough sketch of the region R.

(a) (8pts) Find the volume of the solid obtained by rotating the region around the x-axis.

$$V = \int_1^2 \pi (3-x)^2 dx = \pi \int_1^2 (9 - 6x + x^2) dx$$

$$= \pi \left(9x - 3x^2 + \frac{x^3}{3}\right|_1^2$$

$$= \pi \left(18 - 12 + \frac{8}{3} - 9 + 3 - \frac{1}{3}\right) = \frac{7\pi}{3}$$

(b) (8pts) Find the volume of the solid obtained by rotating the region around the y-axis.

$$V = \int_1^2 2\pi x (3-x) \, dx = 2\pi \int_1^2 (3x - x^2) dx$$

$$= 2\pi \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right|_1^2 = \frac{13\pi}{3}$$

$$= 2\pi \left(\frac{3}{2} \cdot 4 - \frac{8}{3} - \frac{3}{2} + \frac{1}{3}\right) = 2\pi\left(\frac{36}{6} - \frac{16}{6} - \frac{9}{6} + \frac{2}{6}\right)$$

10. (8pts) Consider the region S bounded by the curves $y = x^2$ and $y^2 = 27x$ sketched in Figure B below. Each integral below is the volume of a solid obtained by rotating S around a particular axis. Match the correct axis with the expression for volume by writing the appropriate letter in the blank provided. Each answer is used exactly once.

D $\int_0^3 \pi((10 - x^2)^2 - (10 - \sqrt{27x})^2) \, dx$ — A. x-axis

C $\int_0^3 2\pi(1 + x)(\sqrt{27x} - x^2) \, dx$ — B. y-axis

A $\int_0^3 \pi(27x - x^4) \, dx$ — C. $x = -1$

B $\int_0^9 \pi(y - (\frac{y}{27})^2) \, dy$ — D. $y = 10$

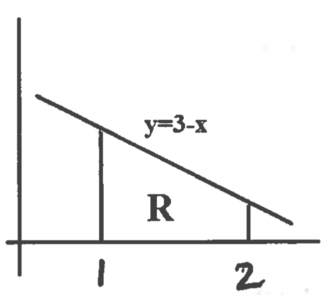


Figure A

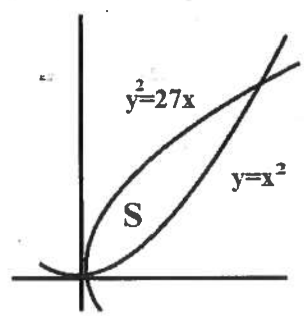


Figure B

11. (8pts) Find the length of the arc of the curve $y = \frac{2}{3}(x+2)^{3/2}$ from $x = -2$ to $x = 1$.

$$y' = (x+2)^{1/2} \Rightarrow (y')^2 = x+2$$

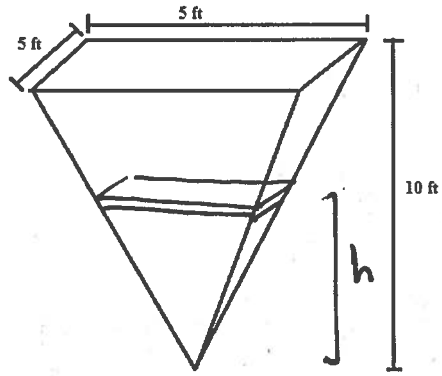
$$L = \int_{-2}^{1} \sqrt{1+(x+2)}\,dx = \int_{-2}^{1} \sqrt{x+3}\,dx = \int_{1}^{4} u^{1/2}\,du = \frac{2}{3} u^{3/2} \Big|_1^4$$

$u = x+3$

$du = dx$

$$= \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(7) = \frac{14}{3}$$

12. (10pts) A tank in the shape of an inverted square pyramid is filled with water. The tank is 10 feet tall and has sides of 5 feet at its rim. How much work is required to pump the water over the top edge of the tank? Use ρ to denote the density of water in lbs/ft^3.



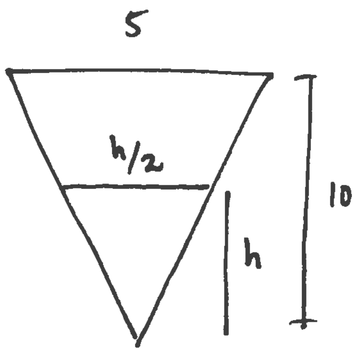
Volume of slice of H_2O at height h:

$$(h/2)^2\,dh$$

Weight of slice of H_2O at height h:

$$\rho\, h^2/4\, dh$$

distance lifted : $10 - h$



$$W = \int_0^{10} \frac{\rho}{4} h^2 (10-h)\,dh$$

$$= \frac{\rho}{4} \int_0^{10} (10h^2 - h^3)\,dh$$

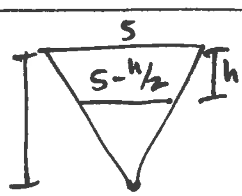
$$= \frac{\rho}{4}\left(\frac{10}{3}h^3 - \frac{h^4}{4}\right)\Big|_0^{10}$$

$$= \frac{\rho}{4}(10)^4\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\rho}{4}(10)^4\left(\frac{1}{12}\right)$$

$$= \frac{10{,}000\rho}{48} \text{ ft}\cdot\text{lbs}$$

Another possibility



$$W = \int_0^{10} \rho (5 - h/2)^2 h\,dh$$