Departement of Mathematics
University of Utah
Real and Complex Analysis Qualifying Exam

Instructions: Do seven problems, at least three from part A and three from part B. List the problems you have done on the front of your blue book.

Part A.

1. Let ℓ^1 be the Banach space of real sequences $a = (a_1, a_2, \dots)$ with norm $\|a\|_1 = \sum_{i=1}^{\infty} |a_i| < \infty$, and ℓ^∞ the Banach space of real sequences with norm $\|x\|_\infty = \sup_i |x_i|$.
   Let $a \in \ell^1$ be a fixed sequence; for any sequence $x \in \ell^\infty$, define a new sequence $T_a(x)$ by
   $$\big(T_a(x)\big)_n = \sum_{i=1}^{n} a_i x_i$$
   (a) Show that this defines a bounded operator $T_a \colon \ell^\infty \to \ell^\infty$.
   (b) Show $\|T_a\| = \|a\|_1$, where the left-hand side denotes the norm of the operator $T_a \colon \ell^\infty \to \ell^\infty$.
2. Let $(\mathcal{M}, X, \mu)$ be a **finite** positive measure space.
   (a) Show that $L^2(X) \subset L^1(X)$, and that the inclusion $i \colon L^2(X) \subset L^1(X)$ has norm $\|i\| = \sqrt{\mu(X)}$.
   (b) Let $\mathcal{M}' \subset \mathcal{M}$ be the subset of measurable sets E with $\mu(E) > 0$. Show that if
   $$L^1(X) = L^2(X),$$
   then
   $$\inf_{E \in \mathcal{M}'} \mu(E) > 0.$$
3. Let H be a Hilbert space with inner product written as $\langle f, g \rangle$, and let $f, g \in H$ be two non-zero elements. Show that $f = zg$ for some $z \in \mathbb{C}^*$ if and only there exists no $h \in H$ with
   $$\langle f, h \rangle = 1 \quad \text{and} \quad \langle g, h \rangle = 0.$$
4. Let $f \colon [0,1] \to \mathbb{R}_{\geq 0}$ be a measurable integrable function $f \in L^1([0,1])$ (with respect to the Lebesgue measure). Show that
   $$\int_{[0,1]} x f(x) dx = \int_{[0,1]} \left(\int_{[y,1]} f(x) dx \right) dy$$
5. Let $T \colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be the translation operator defined by $(T.f)(x) = f(x+1)$. Show that there is no non-zero bounded linear functional $\Phi \colon C_0(\mathbb{R}) \to \mathbb{C}$ invariant under T, i.e. such that Φ satisfies
   $$\Phi(f) = \Phi(Tf).$$

Part B. In the following, D and $\overline{D}$ will denote the open and the closed unit disks, respectively.

6. Compute the integral
$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \, dx.$$

7. Let a be an isolated singularity of the meromorphic function $f : \Omega \to \mathbb{C}$. Prove that if a is an essential singularity, then in any neighborhood of a, the function f takes values arbitrarily close to any complex number.

8. Let $\alpha > 1$ be arbitrary. Show that the equation
$$\alpha - z - e^{-z} = 0$$
has exactly one solution in the half plane $\{z : \ \mathrm{Re}\ z > 0\}$, and moreover, this solution is real.

9. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected region, and fix $z_0 \in \Omega$. If $\phi : \Omega \to D$ is a conformal map such that $\phi(z_0) = 0$, show that
$$|\phi'(z_0)| = \sup\{|f'(z_0)| : \ f : \Omega \to D \text{ holomorphic}, f(z_0) = 0\}.$$

10. Let f be a function holomorphic on D and continuous on $\overline{D}$. Assume that $|f(z)| = 1$ whenever $|z| = 1$. Show that f can be extended to a meromorphic function on the whole $\mathbb{C}$, with at most finitely many poles.
(*Hint:* Starting with the Schwarz reflection principle for the upper half plane, deduce that an appropriate reflection continuation that can be used in this setting is $z \mapsto \frac{1}{\overline{f(1/\overline{z})}}$.)