# Probability Prelim

January 3, 2008

There are 10 problems, of which you should turn in solutions for 6 (your best 6). Each problem is worth 10 points, and 40 points is required for passing.

1. Let $X$ and $Y$ have joint density

$$f(x,y) = g\left(\sqrt{x^2+y^2}\right), \qquad (x,y) \in \mathbf{R}^2,$$

for some function $g$. Show that $Z = X/Y$ has a Cauchy density.

2. (a) State the central limit theorem as well as Kolmogorov's 0-1 law.

(b) For arbitrary events $A_n$, $n \geq 1$, show that $\inf_{n\geq 1} P(A_n) > 0$ implies $P(A_n \text{ i.o.}) > 0$.

(c) Let $X_1, X_2, \ldots$ be i.i.d. mean 0, variance 1 random variables, and put $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$. Show that $\limsup_{n\to\infty} S_n = \infty$ a.s.

3. Construct three random variables $X, Y, Z$ such that

$$E[E[X \mid Y] \mid Z] \neq E[E[X \mid Z] \mid Y]$$

with positive probability.

4. Let $X_1, X_2, \ldots$ be a sequence of integrable random variables defined on the same probability space. Show that $\{X_n\}_{n\geq 1}$ is a submartingale if and only if $E[X_T] \geq E[X_1]$ for all bounded stopping times $T$. (Take the filtration to be the natural one, $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$.)

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. positive random variables. Show that

$$E\left[\frac{\sum_{i=1}^m X_i}{\sum_{i=1}^n X_i}\right] = \frac{m}{n}, \qquad 1 \leq m \leq n.$$

6. Suppose $X_{1,n}, X_{2,n}, \dots$ are i.i.d. with $P\{X_{1,n} = j\} = 1/n$ for $j = 1, \dots, n$. Prove that if

$$T_n := \inf\{k \geq 1 : X_{1,n} + \cdots + X_{k,n} > n\},$$

then $\lim_{n\to\infty} P\{T_n = k\}$ exists for all $k \geq 1$. Compute that limit.

7. Let $X_1, X_2, \dots$ be i.i.d. mean 0, variance 1 random variables and $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$. Show directly that the characteristic function of $S_n/\sqrt{n}$ converges pointwise to the characteristic function of the standard normal distribution. Notice that this is a key step in the proof of the central limit theorem, so you are *not* allowed to use the central limit theorem in your derivation.

8. Let $X_1, X_2, \dots$ be i.i.d. with $P(X_1 > x) = e^{-x}$, $x > 0$. Prove that $\limsup_{n\to\infty} X_n / \ln n = 1$ a.s.

9. Let $X$ and $Y$ be independent, each having the standard normal distribution, and let $(R, \Theta)$ be the polar coordinates for $(X, Y)$.

(a) Show that $X + Y$ and $X - Y$ are independent, and that $R^2 = [(X + Y)^2 + (X - Y)^2]/2$, and conclude that the conditional distribution of $R^2$ given $X - Y$ is the chi-squared with one degree of freedom translated by $(X - Y)^2/2$.

(b) Show that the conditional distribution of $R^2$ given $\Theta$ is chi-squared with two degrees of freedom.

(c) If $X - Y = 0$, the conditional distribution of $R^2$ is chi-squared with one degree of freedom. If $\Theta = \pi/4$ or $\Theta = 5\pi/4$, the conditional distribution of $R^2$ is chi-squared with two degrees of freedom. But the events $\{X - Y = 0\}$ and $\{\Theta = \pi/4\} \cup \{\Theta = 5\pi/4\}$ are the same. Resolve the apparent contradiction.

10. Show that the number of fixed points in a random permutation of length $n$ is asymptotically Poisson with mean 1. Hint: Use characteristic functions. Write $e^{it1_A}$ as $1 + (e^{it} - 1)1_A$.