Preliminary Exam, Numerical Analysis, January 2018

Instructions: This exam is closed book, no notes, and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1. (Projections)
Let $P \in \mathbb{C}^{N\times N}$ be a projection matrix. Prove that $\ker(P) \perp \operatorname{range}(P)$ if and only if $P = P^*$. (I.e., you are being asked to justify the definition of an orthogonal projection matrix.)

Problem 2. (Hadamard's inequality)
Let $A \in \mathbb{C}^{N\times N}$ be comprised of columns $a_i \in \mathbb{C}^N$, $i = 1, \ldots, N$. Prove that

$$|\det A| \leq \prod_{i=1}^{N} \|a_i\|_2.$$

You may use, for example, a QR decomposition/Gram-Schmidt argument.

Problem 3. (Quadrature)
Consider the quadrature rule

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_{-1}f(-1) + w_0 f(0) + w_1 f(1) + w_0' f'(0).$$

Compute weights w_{-1}, w_0, w_1, and w_0' such that this quadrature rule is exact for polynomials up to degree 3.

Problem 4. (Finite difference formulas)
Given $h > 0$, compute weights w_j for the following 5-point finite difference formula for the *third* derivative $f^{(3)}(x)$:

$$\begin{aligned} f^{(3)}(x) &\approx \sum_{j=-2}^{2} w_j f(x + jh) \\ &= w_{-2}f(x-2h) + w_{-1}f(x-h) + w_0 f(x) + w_1 f(x+h) + w_2 f(x+2h). \end{aligned}$$

What is the order of accuracy of your formula?

Problem 5. (Absolute stability for ODEs)
For the ODE $y' = f(y)$, and given $p > 0$, a linear multistep scheme has the form

$$\sum_{j=0}^{p} a_j y_{n+j} + \Delta t \sum_{j=0}^{p} b_j f(y_{n+j}) = 0$$

where Δt is the timestep, $a_p = 1$, and y_n is the numerical approximation to $y(t = n\Delta t)$, where the starting time is $t = 0$.

a. Define the region of absolute stability for the above scheme.
b. Compute and sketch the region of absolute stability for the backward (i.e., implicit) Euler scheme.

Problem 6. (Trapezoid rule)
For the ODE $y' = f(t, y)$, the Trapezoid rule is

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{2} \left(f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right).$$

Compute the region of absolute stability for the Trapezoid rule. Is this method A-stable?

Problem 7. (Unconditionally stable scheme)
For the one-dimensional heat equation,

$$u_t = u_{xx},$$

consider the Crank-Nicolson, central difference scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{2} \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \right)$$

where Δt and h are temporal and spatial stepsizes, respectively, and where u_j^n is the numerical approximation to $u(x = jh, t = n\Delta t)$, with $x_0 = 0$ and the initial time $t = 0$. Use von Neumann stability analysis to show that this scheme is unconditionally stable.

Problem 8. (Central scheme)
Consider the one-dimensional advection equation,

$$u_t = cu_x,$$

where $c \in \mathbb{R}$ is the wavespeed. Use von Neumann stability analysis to determine the values of c for which the following the central difference, forward Euler scheme is stable:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c \frac{u_{j+1}^n - u_{j-1}^n}{2h}.$$