DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to have at least 3 completely correct solutions in **both** parts.

A. Answer all of the following questions.

1. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ be the two sphere in $\mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x(y - 2)$.
   (a) Show that 0 is a critical value of f and find all critical points in $\mathbb{R}^3$.
   (b) Show that 0 is a regular value of f restricted to S^2 and conclude that $M = f^{-1}(0) \cap S^2$ is manifold. What manifold is M?
   (c) Define $g : \mathbb{R}^3 \to \mathbb{R}$ by $g(x, y, z) = x + z$. Calculate the critical points of g restricted to S^2 and the critical points of g restricted to M.

2. Define $r : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2$ by
$$r(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$
Calculate $r^*(-ydx + xdy)$ and $r^*(xdx + ydy)$. Show that for any two form ω on $\mathbb{R}^2$, $r^*\omega = 0$ and conclude that for any 1-form α on $\mathbb{R}^2$, $r^*\alpha$ is closed (even if α is not).

3. For $t \in (0, 1)$ define a map $f_t : S^2 \to S^2$ by
$$f_t(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2 + (z + t)^2}}, \frac{y}{\sqrt{x^2 + y^2 + (z + t)^2}}, \frac{z + t}{\sqrt{x^2 + y^2 + (z + t)^2}} \right).$$
Show that f_t is a Lefschetz map with isolated fixed points at the two antipodal points with $z = \pm 1$ and the Lefschetz number is $L(f) = 2$. Conclude that every diffeomorphism of S^2 that is homotopic to the identity has a fixed point. (Hint: One can simply compute the derivative of f_t to see that on the tangent spaces of the two fixed points it is a multiple of the identity. However, if you use that f_t is a composition of the map $(x, y, z) \mapsto (x, y, z+t)$ and the 3-dimensional version of the map r from (2) then the computation is easier.)

4. Let $f : M \to N$ be a differentiable map between n-manifolds and assume that M is compact and N is connected. If $x \in N$ is a regular value of f show that $f^{-1}(x)$ is finitely many points in M. If every point in N is a regular value of f define a function $\# : N \to \mathbb{Z}$ by letting $\#(x)$ be the number of points in $f^{-1}(x)$ and show that $\#$ is constant.

5. Let $f : M \to N$ be a differentiable map and V and W vector fields on M and N, respectively, with $f_*(x)V(x) = W(f(x))$ for all $x \in M$. Assume that ϕ_t and ψ_t are the respective flows of V and W and show that $f \circ \phi_t = \psi_t \circ f$.

6. State Stokes' theorem and use it to give an example (with proof) of a manifold with a form that is closed but not exact.

B. Answer all of the following questions.

7. Regard the n-simplex Δ^n as a Δ-complex in the natural way, with $(n+1)$ vertices and a single n-simplex. Find all subcomplexes $X \subset \Delta^n$ such that $\tilde{H}_{n-1}(X) \neq 0$ (with a proof that there are no others).

8. Let $p, q \geq 2$ be integers and denote by $X_{p,q}$ the join of discrete sets $\{v_1, \cdots, v_p\}$ and $\{w_1, \cdots, w_q\}$. Thus $X_{p,q}$ is a graph with $p+q$ vertices and pq edges.
   (a) What is the fundamental group of $X_{p,q}$?
   (b) Denote by $Y_{p,q}$ the 2-complex obtained from $X_{p,q}$ by attaching 2-cells with attaching maps $S^1 \to X_{p,q}$ homeomorphisms to all possible cycles of length 4. (Thus there are $\binom{p}{2}\binom{q}{2}$ 2-cells.) Show that $Y_{p,q}$ is simply connected.

9. State the Path Lifting Theorem for covering spaces and sketch a proof of it.

10. Let M, N be two closed connected oriented n-manifolds and $f : M \to N$ a degree 1 map. (Recall that this means that $f_*[M] = [N]$ where $[M], [N]$ are the fundamental classes of M, N respectively.)
    (a) Show that $f_* : H_k(M) \to H_k(N)$ is onto for every k. Hint: $f_*(\alpha \cap f^*(z)) = f_*(\alpha) \cap z$. Explain.
    (b) Deduce that if $M = S_g$, $N = S_h$ are surfaces of genus g, h respectively, then $g \geq h$.

11. Let X be a connected surface with nonempty boundary ∂X and $p \in X$. Show that inclusion
$$X \setminus \{p\} \hookrightarrow X$$
induces an isomorphism in H_1 if and only if $p \in \partial X$. You may use the fact that $H_2(X) = 0$ without proof.

12. Prove that the fundamental group of any Lie group is abelian.