DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to pass **both** parts.

A. Answer all of the following questions.

1. Let A be an element of the Lie algebra of $\mathrm{GL_n}(\mathbb{R})$. What is the left-invariant vector field on $\mathrm{GL_n}(\mathbb{R})$ that is associated to A? Show that it is left-invariant.

2. For $t \in \mathbb{R}$, find the matrix
$$\exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

3. The cup product for the de Rham cohomology of a smooth manifold M is defined as
$$\cup : H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$$
where $[\omega] \cup [\theta] = [\omega \wedge \theta]$. Show that the cup product is well-defined.

4. For $(x, y) \in \mathbb{R}^2 - \{(0,0)\}$, let
$$\omega(x,y) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$
Find $\int_{C(r)} \omega$ where $C(r)$ is the circle of vectors in $\mathbb{R}^2$ of norm equal to $r > 0$ oriented counterclockwise. Find a smooth function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ such that $\omega = df$ or explain why there is no such f.

5. Is the plane field on $\mathbb{R}^3$ that is spanned at the point (x, y, z) by $2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$ integrable? Explain.

6. Let G be a Lie group, and let $\widetilde{G}$ be the universal cover of G. Endow $\widetilde{G}$ with the structure of a Lie group.

7. Let G be a Lie group, and H a closed Lie subgroup. Sketch the proof that G/H is a manifold.

B. Answer all of the following questions.

8. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$. Give an example of an irregular covering space $\tilde{X} \to X$. You should draw the picture of the 1-skeleton of $\tilde{X}$ (make sure 2-cells lift).

9. Compute the fundamental group of the space obtained from the disjoint union $A \sqcup B$ of two 2-tori by identifying them along 3 points, i.e. $a_i \sim b_i$ for $i = 1, 2, 3$ where $\{a_1, a_2, a_3\} \subset A$ and $\{b_1, b_2, b_3\} \subset B$.

10. Let M be a closed connected orientable 3-manifold with $H_1(M;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Compute $H_i(M;\mathbb{Z})$ and $H^i(M;\mathbb{Z})$ for all i.

11. Let S_g be the closed connected oriented surface of genus g.
    (a) Prove that for $g < h$ every map $S_g \to S_h$ has degree 0.
    (b) Prove or disprove: for $g < h$ every map $S_g \to S_h$ is nullhomotopic?

12. Let $X = S^1 \times [0,1]$ and $\partial X = S^1 \times \{0,1\}$. Compute $H_i(X, \partial X; \mathbb{Z})$ for all i.

13. (a) The complex projective space $\mathbb{C}P^n$ can be obtained from $\mathbb{C}P^{n-1}$ by attaching a single $2n$-cell. Describe the attaching map.
    (b) What is $H^k(\mathbb{C}P^n;\mathbb{Z})$? Give a proof by induction on n using (a). State any theorems you are using.
    (c) What is the ring structure on $H^*(\mathbb{C}P^n;\mathbb{Z})$? Give an inductive proof. You may use Poincaré duality.

14. Carefully state the van Kampen theorem.