DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to have at least 3 completely correct solutions in part A along with passing part B. If you don't pass B but get 4 problems from part A correct you will have passed that section of the exam.

A. Answer all of the following questions.

1. (a) State the definition of a regular value and state the pre-image theorem.
   (b) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function with $y \in \mathbb{R}^m$ a regular value and let $M = F^{-1}(y)$. If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function and $x \in M$ is critical point for $f|_M$ what are the possible values for the dimension of $\ker F_*(x) \cap \ker f_*(x)$? (Here $F_*(x)$ is the tangent map from $T_x\mathbb{R}^n \to T_y\mathbb{R}^m$ and $f_*(x)$ is also a tangent map.)
   (c) Define $G : \mathbb{R}^n \to \mathbb{R}^{m+k}$ by $G(x) = (F(x), f(x))$.
      (i) If $z \in \mathbb{R}^k$ is regular value of $f|_M$ is $(y, z) \in \mathbb{R}^{m+k}$ always a regular value of G? Give a proof or find a counterexample.
      (ii) If $z \in \mathbb{R}^k$ is a regular of f (as a function on $\mathbb{R}^n$) is $(y, z) \in \mathbb{R}^{m+k}$ always a regular value of G. Give a proof or find a counterexample.

2. Let M be a differentiable manifold. Prove that its tangent bundle TM and its cotangent bundle TM^* are isomorphic as smooth vector bundles.

3. Let V be a smooth vector field on $\mathbb{R}^2$ and assume that outside of a compact set $V = \frac{\partial}{\partial x}$. Show that the flow for V is defined for all time.

4. (a) State Stokes theorem.
   (b) If $\omega \in \Omega^n(\mathbb{R}^n)$ has compact support and $\int_{\mathbb{R}^n} \omega \neq 0$ show that there does not exist an $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ with compact support and $d\alpha = \omega$.
   (c) Now assume that $n = 1$ and that $\int_{\mathbb{R}} \omega = 0$. Find an $\alpha \in \Omega^0(\mathbb{R})$ with compact support and $d\alpha = \omega$.

5. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2\}$. Define $\pi : \mathbb{R}^3 \backslash \{0\} \to S^2$ by $\pi(x, y, z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ and $T_\epsilon(x, y, z) = (x, y, z + \epsilon)$. For $\epsilon \in (0, 1)$ the map $f_\epsilon = \pi \circ T_\epsilon$ is a Lefschetz map from S^2 to itself. Calculate its Lefschetz number and conclude every map of S^2 to itself that is homotopic to the identity has a fixed point. (Hint: It will be easier to calculate the derivative of π and T_ϵ separately and use the chain rule to find the derivative of f_ϵ.)

6. Let ω and η be closed forms on a manifold M. Show that the de Rham cohomology class of $\omega \wedge \eta$ only depends on the cohomology classes of ω and η.