PRELIMINARY EXAMINATION IN ALGEBRA

August 18, 2009

Instructions: Answer as many questions or parts of questions as you wish. A passing score consists of four complete answers or a reasonable equivalent.

1. Compute the number of elements of order 4 in the symmetric group S_7.
2. Let G be a group of order p^n, where p is a prime number and $n > 0$ is an integer. Prove that the center of G is not trivial.
3. Let $E \subseteq F$ be a finite Galois extension of fields. Suppose that there is an element α in F such that $\alpha \notin E$ and such that α is in every proper extension of E contained in F. Show that the Galois group of F over E is cyclic of prime power order.
4. Show that every finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.
5. Show that there are no simple groups of order 520.
6. Let M be a matrix with real coefficients such that $M^2 + M + I = 0$. Give the possible canonical forms for M

   (a.) Over the real numbers $\mathbb{R}$ (rational canonical form).

   (b.) Over the complex numbers $\mathbb{C}$ (Jordan canonical form).
7. Let R be a commutative ring, and let S be a multiplicative subset of R ($1 \in S$, and if s and t are in S, then st is in S). Show that an ideal I that is maximal with the property that $S \cap I = \emptyset$ is prime.
8. Determine for which integers q the polynomial $X^3 + 1$ has three distinct roots in the field with q elements.
9. Show that there exists a Galois extension of the field $\mathbb{Q}$ of rational numbers with Galois group $\mathbb{Z}/5\mathbb{Z}$.
10. State and prove the Eisenstein Irreducibility Criterion.