Question 1. Suppose G is a group and S a subset of G. Write down a real proof that $C(G, S)$ is connected if and only if S is a generating set for G. Recall that S generates G If each element of G can be written as a product of elements of S and their inverses.

Question 2. Suppose G is a group and S a (finite? not sure if you need this) generating set of G. Suppose H is a subgroup of G. Show that H is of finite index in G if and only if there exists a $D > 0$ such that each element of g is within D of an element of H.

Question 3. Use the "drawing trick" to draw the Cayley graph of D_n (symmetries of the regular n-gon) with respect to two adjacent reflections.

Question 4. Try to do the "drawing trick" for S_4 generated by the 3 adjacent transpositions $\{(1\,2), (2\,3), (3\,4)\}$ by viewing S_4 as the symmetry group of a regular tetrahedron.

Question 5. Harder problem: Give a combinatorial description of the Cayley graph of S_n with respect to the elementary transpositions $T_{n-1} = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$.

Question 6. Use the fact that $C(S_n, T_{n-1})$ from the previous problem is a bipartite graph to show that each permutation $\sigma \in S_n$ can be assigned a partity in a well-defined way - more rigorously, there is a group homomorphism f from S_n to $\mathbb{Z}_2$ (the group of order 2) defined by $f(\sigma)$ is 0 if σ can be written as a product of an even number of transpositions and $f(\sigma) = 1$ if σ can be written as a product of an odd number of transpositions.

Hints:

- In a bipartite graph, all loops have even length.
- We already know that each $\sigma \in S_n$ can be written as a product of transpositions - but not in a unique way. If you don't know this fact already, I bet you can convince yourself of it pretty quickly - you will use this fact of course.
- Each transposition $(i\ j)$ can be written as the product of an odd number of elements from T_{n-1} - you should prove this. First show it for transpositions of the form $(1\ a)$ for any $a > 1$. Then use the fact that $(a\ b)$ with $a < b$ can be written as $(1\ a)(1\ b)(1\ a)$ to conclude what you need.

Question 7. Draw the Cayley graph of $G = \mathbb{Z} \oplus \mathbb{Z}$ where G is generated by $S = \{(\pm 1, 0), (0, \pm 1)\}$ as an undirected graph with two colors.

Question 8. See if you can explain how to "recognize" a normal subgroup H in $C(G, S)$ where H is generated by a subset of S.